Vorlesung aus dem Sommersemester 2013

Algebraic Topology II

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Contents

1 Singular Homology Theory (Cont.)

1.1 Classical Theorems of Topology (Cont.)

LEMMA 1.1. *Let* $F\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ *be the reflection at a hyperplane* $H \subset \mathbb{R}^{n+1}$*. Then we have in particular* $F \in O(n+1)$ *with* $\det(F) = -1$ *and hence* $F(S^n) \subset S^n$*. Write* $f = F|_{S^n}$*. Then* $\deg(f) = -1$*.*

COROLLARY 1.2.

(i) *For any orthogonal map* $A \in O(n)$ *we have* $\deg(A|_{S^{n-1}}) = \det(A)$*.*
(ii) *The antipodal map* $-\mathrm{id}\colon S^n \longrightarrow S^n$ *has degree* $\deg(-\mathrm{id}) = (-1)^{n+1}$*.*
(iii) *If* $f\colon S^n \longrightarrow S^n$ *has no fixed point then* $\deg(f) = (-1)^{n+1}$*.*

THEOREM 1.3 (Hairy Ball Theorem). *The* n*–sphere* S^n *admits a continuous vector field without zeroes if and only if* n *is odd.*

Proof. If V is such a vector field then we may assume that $\|V(p)\| = 1$ for all $p \in S^n$. Consider the homotopy $H\colon S^n \times [0,1] \longrightarrow S^n$ given by

$$H(p,t) = \cos(t)p + \sin(t)V(p).$$

It is a homotopy from id to $-$id which is impossible unless n is odd because of corollary 1.2.

If $n = 2k-1$ is odd then

$$V(x_1, \ldots, x_{2k}) = (x_2, -x_1, \ldots, x_{2k}, -x_{2k-1})$$

defines a continuous vector field on S^n without zeroes. □

THEOREM 1.4. *If* n *is even then* $\mathbb{Z}^\times$ *is the only nontrivial group which can act freely on* S^n*.*

Proof. Assume that a group G acts freely on S^n. Consider the homomorphism $d\colon G \longrightarrow \mathbb{Z}^\times$ given by $d(g) = \deg(\ell_g)$ where ℓ_g denotes the left translation by g. But ℓ_g has no fixed point for $g \neq e$, hence $\deg(\ell_g) = -1$ by corollary 1.2. This implies $\ker(d) = 1$ and that G embeds into $\mathbb{Z}^\times$; i. e. G is either trivial or $\mathbb{Z}^\times$. □

THEOREM 1.5.

(i) *If* $D \subset S^n$ *is homeomorphic to the closed* k*–disk* D^k *for any* $0 \leqslant k \leqslant n$ *then*

$$\widetilde{H}_i(S^n \setminus D) = 0$$

for all $i \in \mathbb{Z}$*.*

(ii) *If* $S \subset S^n$ *is homeomorphic to* S^k *for any* $0 \leqslant k < n$ *then*

$$\widetilde{H}_i(S^n \setminus S) = \begin{cases} \mathbb{Z}, & i = n-k-1 \\ 0, & \textit{otherwise.} \end{cases}$$

REMARK 1.6.

(i) For $n = 2$ the second part of theorem 1.5 is precisely the classical *Jordan curve theorem.*
(ii) In higher dimensions the complement of an embedded $(n-1)$–sphere in S^n has 2 connected components but in general these are not homeomorphic to balls; contrary to dimension $n = 2$.

The proof of this theorem uses a general tool to calculate homology. Assume a space X may be written as $X = A^\circ \cup B^\circ$ for subsets $A, B \subset X$. Write $i_A \colon A \cap B \hookrightarrow A$, $i_B \colon A \cap B \hookrightarrow B$, $j_A \colon A \hookrightarrow X$ and $j_B \colon B \hookrightarrow X$ for the inclusions. Then there is a short exact sequence

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{(i_A)_* + (i_B)_*} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{(j_A)_* - (j_B)_*} C^{\mathcal{U}}_\bullet(X) \longrightarrow 0$$

where $C^{\mathcal{U}}_\bullet(X)$ denotes the chain complex of $\mathcal{U}$-small chains in X for the cover $\mathcal{U} = \{A, B\}$. Hence, one obtains a long exact sequence

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X) \longrightarrow \cdots$$

in homology; the *Mayer–Vietoris sequence.* Similarly, there is a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_i(A \cap B) \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow \widetilde{H}_i(X) \longrightarrow \cdots$$

in reduced homology.

Proof of theorem 1.5.

(i) For $k = 0$ it is clear that $S^n \setminus D \cong \mathbb{R}^n$. In general, let $h \colon I^k \longrightarrow D$ be a homeomorphism and consider the open sets $A = S^n \setminus h(I^{k-1} \times [0, 1/2])$ and $B = S^n \setminus h(I^{k-1} \times [1/2, 1])$. Then $A \cup B = S^n \setminus h(I^{k-1} \times \{1/2\})$ and $A \cap B = S^n \setminus D$. By induction, we may conclude that $\widetilde{H}_\bullet(A \cup B) = 0$. Mayer–Vietoris now implies that

$$0 \longrightarrow \widetilde{H}_i(S^n \setminus D) \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow 0$$

is exact for all i. Assume there is a nontrivial class $\alpha = [c] \in \widetilde{H}_i(S^n \setminus D)$. Then $(i_A)_* c$ is not a boundary in A or $(i_B)_* c$ is not a boundary in B. By iterating this construction, we obtain nested intervals $I_m \subset I$ of length 2^{-m} for all $m \in \mathbb{N}$ such that c is not a boundary in $S^n \setminus h(I^{k-1} \times I_m)$. Let $\{p\} = \bigcap I_m$. By induction, c is a boundary in $S^n \setminus h(I^{k-1} \times \{p\})$, i. e. $c = \partial b$ for some chain b in $S^n \setminus h(I^{k-1} \times \{p\})$. But then—by compactness—there is some large m_0 such that the support of b avoids $h(I^{k-1} \times I_{m_0})$. Hence, b is actually a chain in $S^n \setminus h(I^{k-1} \times I_{m_0})$ and there $c = \partial b$ is a boundary contrary to our assumptions.

(ii) For $k = 0$ it is again clear that $S^n \setminus S \simeq S^{n-1}$. In general, write $S = D_1 \cup D_2$ with $D_i \cong D^k$ and $D_1 \cap D_2 \cong S^{k-1}$. For $A = S^n \setminus D_1$ and $B = S^n \setminus D_2$ Mayer–Vietoris gives an exact sequence

$$\cdots \longrightarrow \widetilde{H}_{i+1}(A) \oplus \widetilde{H}_{i+1}(B) \longrightarrow \widetilde{H}_{i+1}(A \cup B) \longrightarrow \widetilde{H}_i(A \cap B) \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow \cdots$$

i. e. by induction there is an isomorphism

$$\widetilde{H}_i(A \cap B) = \widetilde{H}_i(S^n \setminus S) \cong \begin{cases} \mathbb{Z} & i + 1 = n - k \\ 0 & \text{otherwise.} \end{cases}$$ □

Theorem 1.7 (Invariance of domain). *If $X \subset \mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^n$ then X itself must be open.*

Proof. Identify $\mathbb{R}^n$ with $S^n \setminus \{N\}$. Then it is enough to show that X is open in S^n. Let $f \colon X \longrightarrow f(X) \subset \mathbb{R}^n$ be a homeomorphism onto an open subset $f(X) \subset \mathbb{R}^n$. Then, for all $x \in X$ there exists a neighbourhood $D \subset X$ of x such that $D \cong D^n$ and $\partial D \cong S^{n-1}$. Theorem 1.5 implies that $S^n \setminus D$ is path-connected and that $S^n \setminus S$ has 2 path-connected components. Hence, $D \setminus S$ is a path-connected component of $S^n \setminus S$ and as such is open because $S^n \setminus S$ is locally path connected. Therefore, $D \setminus S$ is an open neighbourhood of x in S^n which finishes the proof. □

1.2 CW–Complexes and Cellular Homology

Definition 1.8. Let X be a topological space and let $A \subset X$ be a subset. One says that X is obtained from A by *attaching n–cells* if there are maps $\varphi_\alpha \colon S^{n-1}_\alpha \longrightarrow A$ from the standard $(n-1)$–sphere $S^{n-1}_\alpha = S^{n-1}$ such that there is a pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha\in I} S^{n-1}_\alpha & \xrightarrow{\coprod_\alpha \varphi_\alpha} & A \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in I} D^n_\alpha & \xrightarrow[\coprod_\alpha \psi_\alpha]{} & X, \end{array}$$

i. e. such that

$$X = {A \sqcup_{\alpha\in I} D^n_\alpha}\big/{(x \sim \varphi_\alpha(x) \text{ for } x \in \partial D^n_\alpha)}.$$

The *n–cell* e^n_α is the homeomorphic image of $(D^n_\alpha)^\circ$ in X.

Definition 1.9. A topological space X is called *CW–complex* if it is obtained in the following way:
- (i) The 0–*skeleton* $X^{(0)} = \coprod_{\alpha\in I} e^0_\alpha$ is just a discrete topological space.
- (ii) The n–*skeleton* is obtained from X^{n-1} by attaching n–cells.
- (iii) The space X is equal to the union $\bigcup_{n\geqslant 0} X^{(n)}$ carries the “weak topology” or “colimit topology,” i. e. a subset $A \subset X$ is open if and only if $A \cap X^{(n)}$ is open for all $n \geqslant 0$.

Remark 1.10.
- (i) If X is *finite–dimensional*, i. e. $X = X^{(n)}$ for some $n \geqslant 0$, then part (iii) of the definition is automatic.
- (ii) The map $\psi_\alpha \colon D^n_\alpha \longrightarrow X^{(n)} \hookrightarrow X$ is called the *characteristic map* of the n–cell e^n_α. Note that ψ_α induces a homeomorphism between the interiors of D^n_α and e^n_α.
- (iii) The space X is *finite*, i. e. it has only finitely many cells, if and only if X is compact.
- (iv) All CW–complexes are Hausdorff spaces.

Example 1.11.
- (i) The n–sphere S^n is a CW–complex with two cells; one 0–cell, the north pole for instance, and one n–cell, the sphere itself.
- (ii) There are other CW–structures on S^n. For example, one can start with S^0 and in every dimension k attach two k–cells. Here, the k–skeleton is simply the k–sphere S^k. This CW–structure has the advantage of being invariant under the antipodal map. Hence, we get a CW–structure on $\mathbb{RP}^n$ with one cell in each dimension and k–skeleton $\mathbb{RP}^k$.

Definition 1.12. Let X be a CW–complex. A subset $A \subset X$ is called a *subcomplex* if A is the union of cells of X such that if $e^n_\alpha \subset A$ then also $\overline{e^n_\alpha} \subset A$.

Remark 1.13. For each cell $e^n_\alpha \subset A$ in a subcomplex $A \subset X$ one has that $\varphi_\alpha(S^{n-1}_\alpha) \subset A$. In particular A is a CW–complex itself.

Example 1.14. For any CW–complex, the k–skeleton $X^{(k)} \subset X$ is a subcomplex. In particular, $S^k \subset S^n$ and $\mathbb{RP}^k \subset \mathbb{RP}^n$ are subcomplexes.

Lemma 1.15. *Let X be a CW–complex with a compact subset $K \subset X$. Then K is contained in some finite subcomplex of X.*

Proof. The subset K meets only finitely many cells, for suppose there exists a sequence x_i, $i \in \mathbb{N}$ of points in K such that the x_i lie in distinct cells. Then $S = \{x_1, \dots\} \subset K$ is closed: Assume by induction that $S \cap X^{(n-1)}$ is closed. For each n–cell e^n_α the preimage $\varphi^{-1}_\alpha(S) \subset S^{n-1}_\alpha$ is closed and hence $\psi^{-1}_\alpha(S) \subset D^n_\alpha$ is closed. This implies that $S \cap X^{(n)}$ is closed. Similarly, any subset of S is closed, whence S is discrete. Hence, S being also compact, it is finite, contradictory to our assumption.

Now, every finite union of cells is contained in a finite subcomplex: For an n–cell e^n_α, consider $\varphi_\alpha(S^{n-1}_\alpha) \subset X^{(n-1)}$. This is compact, hence by the previous paragraph it is contained in a finite union of cells of dimension at most $n-1$. By induction, we may assume that $\varphi_\alpha(S^{n-1}_\alpha)$ is contained in a finite subcomplex $A \subset X$. Hence, e^n_α is contained in the finite subcomplex $e^n_\alpha \cup A$. Since a finite union of finite subcomplexes is again a finite subcomplex the result follows. □

Proposition 1.16. *Let X be a CW–complex. Then*

(i) *the relative homology $H_k(X^{(n)}, X^{(n-1)})$ is nonzero only for $k = n$ and $H_n(X^{(n)}, X^{(n-1)})$ is the free abelian group on the set of n–cells.*

(ii) *the n–skeleton has no homology in degree $k > n$, i. e. $H_k(X^{(n)}) = 0$ for $k > n$. In particular, if X is finite dimensional then it has no homology in degrees bigger than $\dim X$.*

(iii) *the map $i_* : H_k(X^{(n)}) \longrightarrow H_k(X)$ induced by the inclusion $i : X^{(n)} \hookrightarrow X$ is an isomorphism for $k < n$.*

Proof.

(i) Observe that $(X^{(n)}, X^{(n-1)})$ is a good pair: If $X^{(n)}$ arises from $X^{(n-1)}$ by attaching n–cells D^n_α for $\alpha \in I$ then a neighbourhood which deformation retracts to $X^{(n-1)}$ may be obtained by attaching $D^{(n)} \smallsetminus \{0\}$ for $\alpha \in I$. Also, the quotient $X^{(n)}/X^{(n-1)}$ is just

$$X^{(n)}/X^{(n-1)} \cong \bigvee_{\alpha \in I} S^n_\alpha.$$

We obtain

$$H_k(X^{(n)}, X^{(n-1)}) \cong \widetilde{H}_k\Big(\bigvee_{\alpha \in I} S^n_\alpha\Big) \cong \bigoplus_{\alpha \in I} \widetilde{H}_k(S^n_\alpha)$$

which gives the result.

(ii) Look at the long exact sequence of a pair:

$$\cdots \longrightarrow H_{k+1}(X^{(n)}, X^{(n-1)}) \longrightarrow H_k(X^{(n-1)}) \longrightarrow H_k(X^{(n)}) \longrightarrow H_k(X^{(n)}, X^{(n-1)}) \longrightarrow \cdots$$

Part (i) implies that this sequence gives rise to isomorphisms

$$H_k(X^{(n)}) \xrightarrow{\sim} H_k(X^{(n-1)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H_k(X^{(0)}) = 0$$

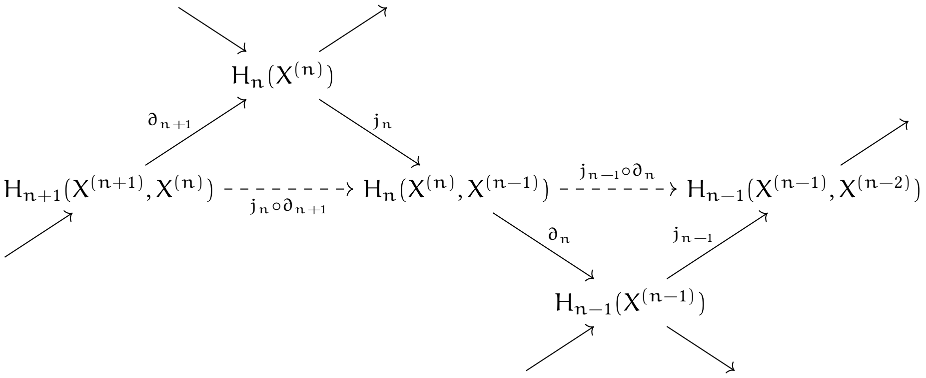
for $k > n \geqslant 0$.

(iii) The same long exact sequence as in (ii) give rise to isomorphisms

$$H_k(X^{(n)}) \xrightarrow{\sim} H_k(X^{(n+1)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} H_k(X^{(n+m)})$$

for $k < n$ and any $m \geqslant 0$ which are induced by the obvious inclusions. This proves (iii) for finite dimensional X. In general, $i_* : H_k(X^{(n)}) \longrightarrow H_k(X)$ is injective: Take $[c] \in H_k(X^{(n)})$ such that $i_*([c]) = 0 \in H_k(X)$, i. e. there exists some $b \in C_k(X)$ with $\partial b = c$. But b has compact support, hence b is chain in $C_k(X^{n+m})$ for some $m \geqslant 0$ because of lemma 1.15. This implies that $[c] = 0 \in H_k(X^{(n+m)})$ and the result for finite dimensional X implies that $[c] = 0$. Surjectivity may be checked analogously. □

Now, look at the long exact sequences of $(X^{(n+1)}, X^{(n)})$, $(X^{(n)}, X^{(n-1)})$ and $(X^{(n-1)}, X^{(n-2)})$:



Write $d_{n+1} = j_n \circ \partial_{n+1}$ and $d_n = j_{n-1} \circ \partial_n$. Then $d_n \circ d_{n+1} = 0$. Hence, we obtain the so called *cellular chain complex* $(C_\bullet^{CW}, d_\bullet)$ where $C_n^{CW} = H_n(X^{(n)}, X^{(n-1)})$ and $d_n = j_{n-1} \circ \partial_n$. By passing to homology we get the *cellular homology groups* $H_n^{CW}(X) = H_n(C_\bullet^{CW}, d_\bullet)$.

Remark 1.17. The reduced homology $\widetilde{H}_n(S^n)$ is *canonically* isomorphic to $\mathbb{Z}$. A preferred isomorphism is given as follows: For $n = 0$ the homology $\widetilde{H}_0(S^0)$ is just the kernel $\ker \varepsilon$ of the augmentation ε and we map $n(x_1 - x_0)$ to n; here $S^0 = \{x_0, x_1\}$. For $n \geqslant 1$ we can use the Mayer–Vietoris sequence for $S^n = D_+^n \cup D_-^n$. It implies that the boundary map $\widetilde{H}_n(S^n) \longrightarrow \widetilde{H}_{n-1}(S^{n-1})$ is an isomorphism.

As a consequence the isomorphism $H_n(X^{(n)}, X^{(n-1)}) \cong \mathbb{Z}^{(I)}$ only depends on the attaching maps $\varphi_\alpha \colon S_\alpha^{n-1} \longrightarrow X^{(n-1)}$.

Proposition 1.18. *There is an isomorphism* $H_n^{CW}(X) \cong H_n(X)$ *for all* $n \in \mathbb{Z}$.

Proof. From the long exact sequence of the pair $(X^{(n+1)}, X^{(n)})$ and proposition 1.16 we have an isomorphism $H_n(X) \cong H_n(X^{(n)})/\operatorname{im} \partial_{n+1}$. On the other hand, because of $H_n(X^{(n-1)}) = 0$ the map j_n is injective, hence $H_n(X) \cong \operatorname{im} j_n / \ker d_{n+1}$. Now, exactness at $H_n(X^{(n)}, X^{(n-1)})$ implies that we have $\operatorname{im} j_n = \ker \partial_n = \ker d_n$. Hence, $H_n(X) \cong \ker d_n / \operatorname{im} d_{n+1} = H_n^{CW}(X)$. □

Consider the commutative diagram

$$\begin{array}{ccc} H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{d_n} & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \bigoplus_{\alpha \in I} \mathbb{Z}_\alpha & \xrightarrow[(d_{\alpha\beta})_{\alpha,\beta}]{} & \bigoplus_{\beta \in J} \mathbb{Z}_\beta \end{array}$$

where I parameterises the n–cells of X and J the $(n-1)$–cells of X, i. e.

$$d_n(e_\alpha^n) = \sum_{\beta \in J} d_{\alpha\beta} e_\beta^{n-1}$$

for $d_{\alpha\beta} \in \mathbb{Z}$.

Lemma 1.19. *The integer* $d_{\alpha\beta}$ *is the degree of the composite map*

$$S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{(n-1)} \twoheadrightarrow \frac{X^{(n-1)}}{(X^{(n-1)} \smallsetminus e_\beta^{n-1})} \cong S_\beta^{n-1}.$$

Proof. Consider the diagram ...

Remark 1.20. There is a canonical injection $\widetilde{H}_k(X) \hookrightarrow H_k(X)$, more precisely, the augmentation $\varepsilon\colon C_0(X) \longrightarrow \mathbb{Z}$ induces a split short exact sequence

$$0 \longrightarrow \widetilde{H}_0(X) \hookrightarrow H_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Remark 1.21.

(i) If X is a CW–complex without n–cells then $H_n(X) = H_n^{CW}(X) = 0$.

(ii) More generally, if X has k n–cells, then $H_n(X)$ is generated by at most k elements.

(iii) In particular, if X is a finite CW–complex, then $H_\bullet(X)$ is finitely generated.

Example 1.22.

(i) Real projective space $\mathbb{RP}^n$ has a cell decomposition $\mathbb{RP}^n = e^0 \cup \cdots \cup e^n$ where the attaching maps $\varphi\colon S^{k-1} \longrightarrow X^{(k-1)} = \mathbb{RP}^{k-1} = S^{k-1}/\mathbb{Z}^\times$ are just the quotient maps. For $n = 2$ the cellular chain complex has the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $d_2 = \deg(S^1 \longrightarrow S^1) = 2$ with $S^1 \longrightarrow S^1$ the nontrivial double cover and the second map is $d_1 = \deg(S^0 \longrightarrow */\emptyset) = 0$. Hence, the homology of $\mathbb{RP}^2$ is

$$\begin{aligned} H_2(X) &= \ker d_2 = 0 \\ H_1(X) &= \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}/2 \\ H_0(X) &= \mathbb{Z}/\operatorname{im} d_1 = \mathbb{Z}. \end{aligned}$$

(ii) Let $X = \Sigma_g$ be the orientable surface of genus g. It is a quotient of a $4g$–gon and has a cellular decomposition with $X^{(1)} = \bigvee_{i=1}^{2g} S^1_i$ and $X^{(2)} = X$. One obtains the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \bigoplus_{a_i, b_i} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

with $d_1 = 0$ and $d_{2,x} = \deg(S^1 \longrightarrow X^{(1)} \longrightarrow X^{(1)}/(X^{(1)} \setminus e^1_x))$ for $x \in \{a_i, b_i\}$. Now, a moment's thought reveals $d_{2,x} = 0$. Hence,

$$\begin{aligned} H_2(\Sigma_g) &= \mathbb{Z} \\ H_1(\Sigma_g) &= \mathbb{Z}^{2g} \\ H_0(\Sigma_g) &= \mathbb{Z}. \end{aligned}$$

(iii) Consider complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1}/\mathbb{C}^\times = S^{2n+1}/S^1$. It is parameterised by homogeneous coordinates $[z_0 : \cdots : z_n]$ and one can always choose z_n to be real and nonnegative. Considering $D^{2n}_+ = \{(z_0, \ldots, z_n) \in S^{2n+1} : z_{n+1} \in \mathbb{R}_{\geqslant 0}\}$ one sees—with the equivalence relation $\sim$ such that $z \sim \lambda z$ for $\lambda \in S^1$ and $z \in \partial D^{2n}_+$—that one can write complex projective space as $\mathbb{CP}^n = D^{2n}_+/\sim\, = \mathbb{CP}^{n-1} \cup e^{2n}$ with attaching map $\varphi\colon S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$ just the quotient map. Inductively one obtains a cell decomposition $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ and hence the homology

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k \text{ even and } 0 \leqslant k \leqslant 2n \\ 0 & \text{otherwise.} \end{cases}$$

Our next goal is to compute the homology $H_\bullet(\mathbb{RP}^n)$. First, we need a tool to compute degrees of maps $f\colon S^n \longrightarrow S^n$ for $n \geqslant 1$. Suppose that there exists some $y \in S^n$ such that the fibre $f^{-1}(y) = \{x_1, \dots, x_m\}$ is finite. Then there exists an open neighbourhoods $V \subset S^n$ of y and $U_i \subset S^n$ of x_i such that the U_i are pairwise disjoint and $f(U_i) \subset V$. In particular, $f(U_i \smallsetminus \{x_i\}) \subset V \smallsetminus \{y\}$. From this we obtain the following commutative diagram:

Since there is a canonical isomorphism $H_n(S^n) \cong \mathbb{Z}$, the groups $H_n(U_i, U_i \smallsetminus x_i)$ and $H_n(V, V \smallsetminus y)$ are also canonically isomorphic to $\mathbb{Z}$. Hence, the map $f_*\colon H_n(U_i, U_i \smallsetminus x_i) \longrightarrow H_n(V, V \smallsetminus y)$ is multiplication with some $\deg f|_{x_i} \in \mathbb{Z}$, the *local degree* of f at x_i.

Example 1.23. If $f\colon S^n \longrightarrow S^n$ is a homeomorphism, then for any $y \in S^n$ the fibre over y has cardinality $\# f^{-1}(y) = 1$ and $\deg f|_{f^{-1}(y)} = \deg f \in \mathbb{Z}^\times$.

Proposition 1.24. *One always has*

$$\deg f = \sum_{i=1}^{n} \deg f|_{x_i}.$$

Proof. By excision the inclusion

$$\coprod_{i=1}^{m} (U_i, U_i \smallsetminus x_i) \hookrightarrow (S^n, S^n \smallsetminus f^{-1}(y))$$

induces isomorphisms on homology

$$\bigoplus_{i=1}^{m} H_n(U_i, U_i \smallsetminus x_i) \xrightarrow{\sim} H_n(S^n, S^n \smallsetminus f^{-1}(y)).$$

We obtain a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^m H_n(U_i, U_i \smallsetminus x_i) & \xrightarrow{\cong} & H_n(S^n, S^n \smallsetminus f^{-1}(y)) \\ \uparrow & \nearrow^{k_i} & \downarrow p_i \\ H_n(U_i, U_i \smallsetminus x_i) & \xrightarrow[\cong]{} & H_n(S^n, S^n \smallsetminus x_i) \end{array}$$

and a short diagram chase reveals

$$j(1) = \sum_{i=1}^{m} k_i(1) \in H_n(S^n, S^n \smallsetminus f^{-1}(y)).$$

On application of f_* one sees

$$\deg f = f_*(j(1)) = \sum_{i=1}^{m} f_*(k_i(1)) = \sum_{i=1}^{m} \deg f|_{x_i}.$$ □

For $\mathbb{RP}^n = e^n \cup \dots \cup e^0$ we obtain the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

and it remains to compute the differential d_k. We have $d_k = \deg f$ for the composition

$$f\colon S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = S^{k-1}.$$

Embedding S^{k-2} into S^{k-1} as the equatorial sphere yields $S^{k-1} \setminus S^{k-2} = (D^{k-1}_+)^\circ \cup (D^{k-1}_-)^\circ$ and f restricts to embeddings $f_\pm \colon (D^{k-1}_\pm)^\circ \longrightarrow S^{k-1}$ such that $f_- = f_+ \circ \tau$ for the antipodal map $\tau = -\mathrm{id}$. Denote by N and S the north and south pole of S^{k-1} respectively. Then

$$\deg f = \deg f|_N + \deg f|_S = 1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd.} \end{cases}$$

Hence, for even n the cellular chain complex looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

and for odd n it looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

Hence, we obtain the homology

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \text{ and } n \text{ odd} \\ \mathbb{Z}/2 & k \text{ odd and } 0 < k < n \\ \mathbb{Z} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For a finite CW–complex X we can define an *Euler–characteristic* of X:

Definition 1.25. The *Euler–characteristic* of X is

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k c_k \in \mathbb{Z}$$

where c_k is the number of k–cells of X.

A priori it is not clear that $\chi(X)$ is a topological invariant of X. However, we have the following result.

Proposition 1.26. *For a finite CW–complex one has*

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rk} H_k(X).$$

The number $b_k = \operatorname{rk} H_k(X)$ *is also called the* k^{th} Betti number *of* X.

Proof. Look at the cellular chain complex

$$0 \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0 \longrightarrow 0$$

where the abelian groups C_k are finitely generated. Write $Z_k = \ker d_k$, $B_k = \operatorname{im} d_{k+1}$ and $H_k = Z_k/B_k$. We obtain short exact sequences

$$0 \longrightarrow Z_k \longrightarrow C_k \xrightarrow{d_k} B_{k-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_k \longrightarrow Z_k \longrightarrow H_k \longrightarrow 0.$$

Now, it is well known that rank of finitely generated abelian groups is additive over short exact sequences. Hence, $\operatorname{rk} C_k = \operatorname{rk} Z_k + \operatorname{rk} B_{k-1}$ and $\operatorname{rk} Z_k = \operatorname{rk} B_k + \operatorname{rk} H_k$. In summary, we obtain

$$\begin{aligned}\chi(X) &= \sum_{k=0}^{\infty} (-1)^k c_k = \\ &= \sum_{k=0}^{\infty} (-1)^k b_k + \sum_{k=0}^{\infty} (-1)^k \operatorname{rk} B_{k-1} + \sum_{k=0}^{\infty} (-1)^k \operatorname{rk} B_k = \\ &= \sum_{k=0}^{\infty} (-1)^k \operatorname{rk} H_k(X).\end{aligned}$$

□

Example 1.27. Denote by Σ_g the closed oriented surface of genus g. From our computation of the homology $H_\bullet(\Sigma_g)$ we immediately obtain $\chi(\Sigma_g) = 2 - 2g$.

2 Homology with Coefficients & Cohomology

2.1 Categories and Functors

Definition 2.1. A *category* $\mathcal{C}$ consists of

(i) a class of objects $\mathrm{Ob}(\mathcal{C})$.

(ii) for each pair (X, Y) of objects of $\mathcal{C}$ a set of *morphisms* $\mathcal{C}(X, Y)$ together with distinguished elements $\mathrm{id}_X \in \mathcal{C}(X, X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$.

(iii) for each triple (X, Y, Z) of objects of $\mathcal{C}$ a composition map

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \xrightarrow{\circ} \mathcal{C}(X, Z)$$

written as $\circ(g, f) = f \circ g$ paralleling composition of functions of sets. This composition is supposed to be associative and to have identities, i. e. $(f \circ g) \circ h = f \circ (g \circ h)$ and $\mathrm{id} \circ f = f = f \circ \mathrm{id}$, whenever these equations make sense.

Remark 2.2. A category $\mathcal{C}$ where $\mathrm{Ob}(\mathcal{C})$ is in fact a set is called *small*. Sometimes our concept of category is called a *locally small category* in the literature, since morphisms are supposed to form sets.

Example 2.3. Examples of categories abound:

(i) There is a category **Set** of sets with functions as morphisms.

(ii) There is a category **Top** whose objects are topological spaces and whose morphisms are continuous maps.

(iii) There is a category Ho(**Top**) whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps.

(iv) There is a category $\mathbf{Top}^2$ whose objects are pairs (X, A) of topological spaces and whose morphisms are maps of pairs.

(v) There is a category $\mathbf{Top}_*$ whose objects are pointed spaces and whose morphisms are basepoint–preserving maps.

(vi) There is a category **Grp** of groups with group homomorphisms.

(vii) There is a category **Ab** of abelian groups with group homomorphisms. This is called a *full subcategory* of **Grp**, it has the same morphisms but fewer objects than **Grp**.

(viii) There is a category k–**Vect** of vector spaces over a field k with k–linear maps.

(ix) There is a category R–**Mod** of R–modules with R–linear maps.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ consists of an assignment $F\colon \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$ together with assignments $F\colon \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$ for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that $F(f \circ g) = Ff \circ Fg$ and $F\,\mathrm{id} = \mathrm{id}$ whenever these equations make sense. If there is no danger of confusion, we write f_* instead of Ff.

Similarly, a *contravariant functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is an assignment on object together with assignments $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FY, FX)$ such that $F(f \circ g) = Fg \circ Ff$ and $F\,\mathrm{id} = \mathrm{id}$. If there is no danger of confusion we write f^* for Ff if F is contravariant.

Example 2.5. Examples of covariant functors abound:

(i) The fundamental group defines a functor $\pi_1\colon \mathbf{Top}_* \longrightarrow \mathbf{Grp}$. Similarly, there are homology functors $H_n\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$ and $H_n\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$.

(ii) For any category $\mathcal{C}$ and any object X there is a functor $\mathcal{C}(X, \_) = \mathrm{Hom}(X, \_)\colon \mathcal{C} \longrightarrow \mathbf{Set}$ such that $\mathrm{Hom}(X, f)(\varphi) = f \circ \varphi$.

(iii) There are various *forgetful* functors obtained by forgetting structure, for example $\mathbf{Top} \longrightarrow \mathbf{Set}$ or $\mathbf{Ab} \longrightarrow \mathbf{Grp}$.

Examples of contravariant functors include:

(i) For a category $\mathcal{C}$ and an object Y there is a functor $\mathcal{C}(\_, Y) = \mathrm{Hom}(\_, Y)\colon \mathcal{C} \longrightarrow \mathbf{Set}$ such that $\mathrm{Hom}(f, Y)(\varphi) = \varphi \circ f$.

(ii) The contravariant functor $\mathrm{Hom}(\_, k)$ in the case of k–**Vect** factors through the category of vector spaces. This is simply the functor mapping a vector space to its dual.

Definition 2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with functors $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$. A *natural transformation* $\eta\colon F \longrightarrow G$ assigns to each object $\mathcal{X} \in \mathrm{Ob}(\mathcal{C})$ a morphism $\eta_X\colon F(X) \longrightarrow G(X)$ such that

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

commutes for all morphisms $f\colon X \longrightarrow Y$. There is of course an analogous definition in the contravariant case.

Example 2.7. For any pair (X, A) the connecting homomorphism $\partial_n\colon H_n(X, A) \longrightarrow H_{n-1}(A)$ in the long exact sequence defines a natural transformation between the two functors F and G defined by $F(X, A) = H_n(X, A)$ and $G(X, A) = H_{n-1}(A)$.

2.2 Homology with Coefficients

Let G be an abelian group. The idea of homology with coefficients is to apply the functor $\_ \otimes G$ to the singular chain complex of a space X and to take homology afterwards. In this sense, one obtains a chain complex with coefficient in G.

Recall, that for abelian groups A and B the *tensor product* $\otimes\colon A \times B \longrightarrow A \otimes B$ satisfies the following universal property: For any $\mathbb{Z}$–bilinear mapping $b\colon A \times B \longrightarrow C$ there exists a unique $\mathbb{Z}$–linear map $\overline{b}\colon A \otimes B \longrightarrow C$ making the diagram

$$\begin{array}{ccc} A \otimes B & \overset{\overline{b}}{\dashrightarrow} & C \\ {\scriptstyle \otimes}\uparrow & \nearrow_{b} & \\ A \times B & & \end{array}$$

commutative. This property characterises the tensor product of A and B up to unique isomorphism. Furthermore, given homomorphisms $\varphi_1\colon A_1 \longrightarrow B_1$ and $\varphi_2\colon A_2 \longrightarrow B_2$ there exists a unique homomorphism $\varphi_1 \otimes \varphi_2\colon A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2$ such that

$$\begin{array}{ccc} A_1 \times A_2 & \xrightarrow{\otimes} & A_1 \otimes A_2 \\ {\scriptstyle \varphi_1 \times \varphi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1 \otimes \varphi_2} \\ B_1 \times B_2 & \xrightarrow[\otimes]{} & B_1 \otimes B_2 \end{array}$$

is commutative. It is immediate that this makes $\_ \otimes \_$ into a bifunctor $\mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$. This functor is symmetric and restricts to functors $\_ \otimes G$ for any abelian group G.

Given a topological space X define the singular chain complex of X with coefficients in G, written $C_\bullet(X; G)$, to be the tensor product $C_\bullet(X) \otimes G$.

Definition 2.8. For an abelian group G the homology $H_n(X; G) = H_n(C_\bullet(X; G))$ is called the n^{th} *singular homology group with coefficients in* G.

Remark 2.9.

(i) It is clear that $C_\bullet(X;\mathbb{Z}) = C_\bullet(X)$ and $H_n(X;\mathbb{Z}) = H_n(X)$.

(ii) Any homomorphism $\varphi\colon G \longrightarrow H$ of abelian groups G and H induces a functorial chain map $\varphi_*\colon C_\bullet(X;G) \longrightarrow C_\bullet(X;H)$, whence a map $\varphi_*\colon H_n(X;G) \longrightarrow H_n(X;H)$ on homology, which is natural with respect to maps $f\colon X \longrightarrow Y$

Of course, there is a corresponding notion of relative homology with coefficients. Define the relative chain complex $C_\bullet(X,A;G) = C_\bullet(X;G)/C_\bullet(A;G)$ with coefficients in G and relative homology $H_n(X,A;G) = H_n(C_\bullet(X,A;G))$. Similarly, one may define reduced homology with coefficients by tensoring the augmented singular chain complex with G, i. e. $\widetilde{H}_n(X;G) = H_n(\widetilde{C}_\bullet(X) \otimes G)$.

Remark 2.10. All the properties of singular homology, e. g. functoriality, the long exact sequence, homotopy invariance, excision, Mayer–Vietoris, carry over to homology with coefficients.

Example 2.11. The computation of the following examples carries over to homology with coefficients:

(i) $\widetilde{H}_\bullet(*;G) = 0$.

(ii) $\widetilde{H}_n(S^n;G) = G$ and $\widetilde{H}_k(S^n;G) = 0$ for $k \neq n$.

Once we have the computation of the reduced homology of spheres available, the development of cellular homology with coefficients works analogously. One defines

$$C_n^{CW}(X;G) = H_n(X^{(n)}, X^{(n-1)};G) \cong \bigoplus_{\alpha\in I} G_\alpha$$

with boundary operators $d_n\colon \bigoplus_\alpha G_\alpha \longrightarrow \bigoplus_\beta G_\beta$ as before. In fact, the matrix elements $(d_n)_{\alpha\beta}$ of these boundary operators are given by the same mapping degrees as before:

Lemma 2.12. *If* $f\colon S^n \longrightarrow S^n$ *has degree* $m \in \mathbb{Z}$*, then* $f_*\colon \widetilde{H}_n(S^n;G) \longrightarrow \widetilde{H}_n(S^n;G)$ *is again multiplication by* m.

Proof. For $g \in G$ consider the unique homomorphism $\varphi\colon \mathbb{Z} \longrightarrow G$ mapping 1 to g. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{H}_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & \widetilde{H}_n(S^n;\mathbb{Z}) \\ {\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \widetilde{H}_n(S^n;G) & \xrightarrow[f_*]{} & \widetilde{H}_n(S^n;G) \end{array}$$

and chase $1 \in \mathbb{Z} \cong \widetilde{H}_n(S^n;\mathbb{Z})$. It follows, that, relative to the identifications $\widetilde{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ and $\widetilde{H}_n(S^n;G) \cong G$, we have

$$f_*(g) = f_*\varphi_*(1) = \varphi_* f_*(1) = \deg(f)g. \qquad \square$$

Example 2.13.

(i) The cellular chain complex of $\mathbb{RP}^n$ with coefficients $\mathbb{Z}/2$ looks like

$$C_\bullet^{CW}(\mathbb{RP}^n, \mathbb{Z}/2) = 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{2=0} \dots \xrightarrow{0} \mathbb{Z}/2 \longrightarrow 0$$

for odd n and like

$$C_\bullet^{CW}(\mathbb{RP}^n, \mathbb{Z}/2) = 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2=0} \mathbb{Z}/2 \xrightarrow{0} \dots \xrightarrow{0} \mathbb{Z}/2 \longrightarrow 0$$

for even n. It follows that

$$H_k(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If k is a field of characteristic $\mathrm{char}(k) \neq 2$, then $2\colon k \longrightarrow k$ is an isomorphism and the cellular chain complex looks like

$$C_\bullet^{CW} = 0 \longrightarrow k \xrightarrow{0} k \xrightarrow{\sim} k \xrightarrow{0} \dots \xrightarrow{0} k \longrightarrow 0$$

for odd n and like

$$C_\bullet^{CW} = 0 \longrightarrow k \xrightarrow{\sim} k \xrightarrow{0} k \xrightarrow{\sim} \dots \xrightarrow{0} k \longrightarrow 0$$

for even n. Hence, we obtain the homology

$$H_m(\mathbb{RP}^n; k) = \begin{cases} k & m = n \text{ odd} \\ k & m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.3 Cohomology

Let G be an abelian group. Apart from taking the tensor product with G we could also apply the contravariant functor $\mathrm{Hom}(\_, G)$ to the singular chain complex $C_\bullet(X)$ of a topological space X. Recall that any group homomorphism $\varphi\colon A \longrightarrow B$ between abelian groups A and B induces a group homomorphism $\varphi^*\colon \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G)$ by precomposition. This is the action of the functor $\mathrm{Hom}(\_, G)$ on morphisms. Define

$$C^n(X; G) = \mathrm{Hom}(C_n(X), G).$$

Elements $\varphi \in C^n(X; G)$ are called *singular* n*–cochains* with coefficients in G and may be identified with a function $\varphi\colon \Delta_n(X) \longrightarrow G$ (because $C_n(X)$ is free on $\Delta_n(X)$). Define the *coboundary map* $\delta^n\colon C^n(X; G) \longrightarrow C^{n+1}(X; G)$ by $\delta^n = \partial_{n+1}^*$. In this way, we obtain the *singular cochain complex* $C^\bullet(X; G)$ with coefficients in G:

$$C^\bullet(X; G) = \dots \longrightarrow C^{n-1}(X; G) \xrightarrow{\delta^{n-1}} C^n(X; G) \xrightarrow{\delta^n} C^{n+1}(X; G) \longrightarrow \dots.$$

The action of δ^n is given more explicitly by

$$\delta^n(\varphi)c = \varphi(\partial_{n+1}c)$$

for every cochain $\varphi \in C^n(X; G)$ and every chain $c \in C^{n+1}(X; G)$. More generally, we call every sequence diagram

$$\dots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \dots$$

with $\delta^n\delta^{n-1} = 0$ a *cochain complex*. Observe that there is really only a difference in notation between cochain complexes and chain complexes. Hence, we will silently use all our results about chain complexes for cochain complexes to; perhaps with the prefix "co" to avoid confusion. In particular, we have the *cohomology* $H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet)$ of a cochain complex $C^\bullet(X)$. Dual to the case of chain complexes, elements of $Z^n = \ker\delta^n$ are called n*–cocycles* and elements of $B^n = \operatorname{im}\delta^{n-1}$ are called n*–coboundaries*.

If $C^\bullet = \mathrm{Hom}(C_\bullet, G)$ arises by dualisation of a chain complex $C_\bullet$, then

- $\varphi \in C^n$ is an n–cocycle if and only if $\delta^n\varphi = 0$, i. e. φ vanishes on $B_n \subset C_n$.
- $\varphi \in C^n$ is an n–coboundary if and only if there is some $\varphi' \in C^{n-1}$ such that $\delta^{n-1}\varphi' = \varphi$.

Definition 2.14. For a topological space X the group $H^n(X; G) = H^n(C^\bullet(X; G))$ is called the n^{th} *singular cohomology group* of X with coefficients in G.

Let's look more closely at a few special cases. For $n = 0$ a 0–cochain $\varphi \in C^0(X; G)$ may be identified with a function $\varphi\colon X \longrightarrow G$. This function is a cocycle if and only if $\varphi(\sigma(1)) = \varphi(\sigma(0))$ for all paths $\sigma\colon I \longrightarrow X$, i. e. φ is constant on path–components. Hence, we obtain the explicit description

$$H^0(X; G) = G^{\pi_0(X)} = \mathrm{Hom}(H_0(X; \mathbb{Z}), G)$$

for the 0^{th} singular cohomology group of X.

We can define a reduced version of cohomology by applying the functor $\mathrm{Hom}(\_, G)$ to the reduced singular chain complex $\widetilde{C}_\bullet(X)$ and taking cohomology. After applying $\mathrm{Hom}(\_, G)$ we have the cochain complex

$$\widetilde{C}^\bullet(X; G) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{\varepsilon^*} C^0(X; G) \xrightarrow{\partial_1^*} C^1(X; G) \xrightarrow{\partial_2^*} \dots$$

where $\varepsilon\colon C_0(X) \longrightarrow \mathbb{Z}$ denotes the augmentation map. Hence, we obtain the *reduced singular cohomology* of X with coefficients in G as

$$\widetilde{H}^k(X; G) = H^k(\widetilde{C}^\bullet(X; G)).$$

In degree $k = 0$ we have $(\varepsilon^*\lambda)(x) = \lambda(1) \in G$ for $x \in X$ and $\lambda \in \mathrm{Hom}(\mathbb{Z}, G)$. Hence, $\operatorname{im}\varepsilon^*$ consists of the constant functions $\varphi\colon X \longrightarrow G$. By identifying G with constant functions $X \longrightarrow G$ we obtain a short exact sequence

$$0 \longrightarrow G \longrightarrow H^0(X; G) \longrightarrow \widetilde{H}^0(X; G) \longrightarrow 0.$$

Similarly, we can define a relative version of cohomology; simply apply $\mathrm{Hom}(\_, G)$ to the relative chain complex $C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$ to obtain the cochain complex $C^\bullet(X, A; G)$ and take cohomology to obtain the *relative singular cohomology* $H^n(X, A; G)$ of a pair (X, A) with coefficients in G. Cochain $\varphi \in C^n(X, A; G)$ may be identified with cochains $\varphi \in C^n(X; G)$ vanishing on $C_n(A)$. In fact, $C^\bullet(X, A; G) \subset C^\bullet(X; G)$ is a subcomplex.

Applying the functor $\mathrm{Hom}(\_, G)$ to the tautological short exact sequence

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i_*} C_\bullet(X) \xrightarrow{j_*} C_\bullet(X, A) \longrightarrow 0$$

we obtain an *exact* sequence of cochain complexes

$$0 \longrightarrow C^\bullet(X, A; G) \xrightarrow{j^*} C^\bullet(X; G) \xrightarrow{i^*} C^\bullet(A; G) \longrightarrow 0$$

because the original sequence splits degreewise. Hence, there is a long exact sequence

$$\dots \longrightarrow H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta^n} H^{n+1}(X, A; G) \longrightarrow \dots$$

A similar calculation gives the analogous sequence for reduced cohomology. In particular, for a pair $(X, \{x_0\})$ there is a long exact sequence

$$\dots \longrightarrow \widetilde{H}^n(\{x_0\}; G) \longrightarrow \widetilde{H}^n(X; G) \longrightarrow H^n(X, \{x_0\}; G) \longrightarrow \widetilde{H}^{n+1}(\{x_0\}; G) \longrightarrow \dots$$

But $H^\bullet(\{x_0\}; G) = 0$ whence there are isomorphisms $\widetilde{H}^n(X; G) \xrightarrow{\sim} \widetilde{H}^n(X, \{x_0\}; G)$ for all n.

Given a map $f\colon X \longrightarrow Y$, applying $\mathrm{Hom}(\_, G)$ to the induced homomorphism $f_*\colon C_\bullet(X) \longrightarrow C_\bullet(Y)$ and taking cohomology gives a morphism $f^*\colon H^n(Y; G) \longrightarrow H^n(X; G)$ for all n. It is immediate that this defines a contravariant functor $H^n(\_; G)\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$. Similarly, singular cohomology of pairs defines a contravariant functor $H^n(\_, \_; G)\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$.

These functors are homotopy invariant, as in the case of singular homology, because a homotopy between maps $f, g\colon X \longrightarrow Y$ induces a chain homotopy between f_* and g_*. This chain homotopy is preserved when applying additive functors, e. g. $\mathrm{Hom}(\_, G)$, whence the induced maps f^* and g^* on cohomology coincide. One may argue analogously for homotopies between maps of pairs.

Singular cohomology also satisfies an excision property: for a pair (X, A) of topological spaces and a subspace $B \subset A$ such that $\overline{B} \subset A^\circ$, the inclusion $i\colon (X \setminus B, A \setminus B) \longrightarrow (X, A)$ induces isomorphisms $i^*\colon H^n(X, A; G) \longrightarrow H^n(X \setminus B, A \setminus B; G)$ for all n. This is shown by simply dualising the proof of excision for singular homology.

In a similar spirit, cohomology admits Mayer–Vietoris sequences. For subspace $A, B \subset X$ such that $X = A^\circ \cup B^\circ$ the inclusions $i_A\colon A \cap B \longrightarrow A$, $i_B\colon A \cap B \longrightarrow B$, $j_A\colon A \longrightarrow X$ and $j_B\colon B \longrightarrow X$ induce a short exact sequence of chain complexes

$$0 \longrightarrow C^\bullet_{\mathcal{U}}(X; G) \xrightarrow{j_A^* - j_B^*} C^\bullet(A; G) \oplus C^\bullet(B; G) \xrightarrow{i_A^* + i_B^*} C^\bullet(A \cap B; G) \longrightarrow 0$$

where $\mathcal{U}$ denotes the cover $\{A, B\}$ of X. Hence, there is a long exact sequence

$$\dots \longrightarrow H^n(X; G) \xrightarrow{j_A^* - j_B^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i_A^* + i_B^*} H^n(A \cap B; G) \xrightarrow{\delta^n} H^{n+1}(X; G) \longrightarrow \dots$$

in cohomology.

3 Homological Algebra

One might naively think that the functors $\mathrm{Hom}(\_, G)$ and $\_ \otimes G$ commute with taking homology, i. e. that $H^n(X; G) = \mathrm{Hom}(H_n(X); G)$ and $H_n(X; G) = H_n(X) \otimes G$, and that we have gained nothing new by introducing these variants of singular homology. But in general this is far from true. Consider for example the cellular chain complex $C_\bullet$ of $\mathbb{RP}^2$:

$$C_\bullet = 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Then $H_2(C_\bullet) = 0$, $H_1(C_\bullet) = \mathbb{Z}/2$ and $H_0(C_\bullet) = \mathbb{Z}$. But after applying the functor $\_ \otimes \mathbb{Z}/2$ to $C_\bullet$ we obtain the chain complex

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow 0$$

and the homology $H_k(C_\bullet \otimes \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leqslant k \leqslant 2$. Similarly, applying $\mathrm{Hom}(\_, \mathbb{Z})$ we obtain the cochain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$$

whence

$$H^k(\mathrm{Hom}(C_\bullet, \mathbb{Z})) = \begin{cases} \mathbb{Z}/2 & k = 2 \\ 0 & k = 1 \\ \mathbb{Z} & k = 0. \end{cases}$$

3.1 The Universal Coefficient Theorem for Homology

Lemma 3.1. *The functor* $\_ \otimes G \colon \mathbf{Ab} \longrightarrow \mathbf{Ab}$ *is right exact, i. e. if*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact, then so is

$$A \otimes G \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes G \xrightarrow{\beta \otimes \mathrm{id}} C \otimes G \longrightarrow 0.$$

Proof. Exactness at $C \otimes G$ is clear. Any additive functor preserves complexes, which of course implies $\mathrm{im}(\alpha \otimes \mathrm{id}) \subset \ker(\beta \otimes \mathrm{id})$. Hence, $\beta \otimes \mathrm{id}$ factors through $B \otimes G/\mathrm{im}(\alpha \otimes \mathrm{id})$:

$$\begin{array}{ccc} B \otimes G & \xrightarrow{\beta \otimes \mathrm{id}} & C \otimes G \\ \big\downarrow & \nearrow_{\psi} & \\ B \otimes G/\mathrm{im}(\alpha \otimes \mathrm{id}) & & \end{array}$$

In fact, ψ is an isomorphism if and only if $\mathrm{im}(\alpha \otimes \mathrm{id}) = \ker(\beta \otimes \mathrm{id})$. The construction of an inverse $\varphi \colon C \otimes G \longrightarrow B \otimes G/\mathrm{im}(\alpha \otimes \mathrm{id})$ may be done as follows. Define

$$\varphi(c \otimes g) := [b \otimes g] \in B \otimes G/\mathrm{im}(\alpha \otimes \mathrm{id})$$

where $\beta(b) = c$. It is enough to check that this is in fact well–defined. If $\beta(b') = \beta(b) = c$, then $\beta(b' - b) = 0$. Hence, $b' - b = \alpha(a)$ for some $a \in A$. But then $(b - b') \otimes g = \alpha(a) \otimes g \in \mathrm{im}(\alpha \otimes \mathrm{id})$ and $[b \otimes g] = [b' \otimes g] \in B \otimes G/\mathrm{im}(\alpha \otimes \mathrm{id})$. □

Lemma 3.2. *Split exact sequences are preserved by any additive functor, in particular they are preserved by* $\_ \otimes G \colon \mathbf{Ab} \longrightarrow \mathbf{Ab}$.

Let $C_\bullet$ denote a chain complex of free abelian groups. There are short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

which induce a short exact sequence

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

where $Z_\bullet$ and $B_{\bullet-1}$ carry the trivial differential. Note that the former sequence splits for all n since B_{n-1} is free (as a subgroup of a free abelian group). But the corresponding sequence of chain complexes is not split in general.

Lemma 3.3. *In the situation above,*

$$0 \longrightarrow Z_\bullet \otimes G \longrightarrow C_\bullet \otimes G \longrightarrow B_{\bullet-1} \otimes G \longrightarrow 0$$

is a short exact sequence of chain complexes.

We obtain a long exact sequence

$$\ldots \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C_\bullet \otimes G) \longrightarrow B_{n-1} \otimes G \xrightarrow{\partial_n} Z_{n-1} \otimes G \longrightarrow \ldots$$

in homology where the connecting homomorphism ∂_n is just $i_{n-1} \otimes \mathrm{id}$ for the inclusion i_{n-1} of B_{n-1} into Z_{n-1}. This long exact sequence may be broken up into short exact sequences

$$0 \longrightarrow \operatorname{coker}(i_n \otimes \mathrm{id}) \longrightarrow H_n(C_\bullet \otimes G) \longrightarrow \ker(i_{n-1} \otimes \mathrm{id}) \longrightarrow 0.$$

Look at the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C_\bullet) \longrightarrow 0. \tag{$*$}$$

Applying $\_ \otimes G$ shows that

$$0 \longrightarrow \ker(i_n \otimes \mathrm{id}) \longrightarrow B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G \longrightarrow H_n(C_\bullet) \otimes G \longrightarrow 0$$

is exact. In particular, $\operatorname{coker}(i_n \otimes \mathrm{id}) = H_n(C_\bullet) \otimes G$. Hence, we have identified $H_n(C_\bullet \otimes G)$ as an extension of $\ker(i_{n-1} \otimes \mathrm{id})$ by $H_n(C_\bullet) \otimes G$. In particular, in the case $\ker(i_{n-1} \otimes \mathrm{id}) = 0$ we have shown $H_n(C_\bullet \otimes G) = H_n(C_\bullet) \otimes G$. Our next goal will be to compute $\ker(i_{n-1} \otimes \mathrm{id})$ in terms of $H_n(C_\bullet)$ and G.

Since B_n and Z_n are free abelian groups the short exact sequence ($*$) is a free resolution of $H_n(C_\bullet)$:

Definition 3.4. For an abelian group H an exact sequence

$$\ldots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

with free abelian groups F_n is called a *free resolution* of H. We generally set $F_{-1} = H$ and extend by 0 to the right to obtain a chain complex $F_\bullet$.

REMARK 3.5. Any abelian group H has a free resolution of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0.$$

Tensoring any free resolution $F_\bullet$ of H with G yields a chain complex

$$\ldots \longrightarrow F_2 \otimes G \xrightarrow{f_2 \otimes \mathrm{id}} F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}} F_0 \otimes G \xrightarrow{f_0 \otimes \mathrm{id}} H \otimes G \longrightarrow 0.$$

DEFINITION 3.6. We define

$$\mathrm{Tor}_n(H, G) \coloneqq H_n(F_\bullet \otimes G).$$

REMARK 3.7.

(i) With our definition, $\mathrm{Tor}_0(H, G) = 0$.

(ii) Under the assumption that Tor_n does not depend on the chosen free resolution, the previous remark implies $\mathrm{Tor}_n(H, G) = 0$ for $n \geqslant 2$. For this reason, we will often write $\mathrm{Tor}(H, G)$ instead of $\mathrm{Tor}_1(H, G)$.

LEMMA 3.8. *Let $F_\bullet$ and $F'_\bullet$ be free resolutions of abelian group H and H'. Then any group homomorphism $\varphi\colon H \longrightarrow H'$ extends to a chain map $\varphi_\bullet\colon F_\bullet \longrightarrow F'_\bullet$ such that $\varphi_{-1} = \varphi$. Furthermore, any two such extensions are chain homotopic.*

Proof. It is clear how to define φ_0 on a basis of F_0 such that

$$\begin{array}{ccccc} F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi = \varphi_{-1}} & & \\ F'_0 & \xrightarrow[f_{0'}]{} & H' & \longrightarrow & 0 \end{array}$$

is commutative. Similarly, if $x \in F_1$ is an element of a basis of F_1, choose some $x' \in F'_1$ such that $f'_1(x') = \varphi_0(f_1(x))$ and set $\varphi_1(x) = x'$. This is possible, because $f'_0\varphi_0 f_1(x) = \varphi f_1 f_0(x) = 0$. Inductively, we obtain the required chain map $\varphi_\bullet$.

Now, suppose that $\varphi'_\bullet$ is another extension of φ to a chain map $F_\bullet \longrightarrow F'_\bullet$. Then $\psi_\bullet = \varphi_\bullet - \varphi'_\bullet$ extends $0\colon H \longrightarrow H'$ and it is enough to show that $\psi_\bullet$ is chain homotopic to 0. To define a chain homotopy $h\colon F_\bullet \longrightarrow F'_{\bullet+1}$, first set $h_k = 0$ for $k \leqslant -1$. If $x \in F_0$ is an element of a basis of F_0, choose $x' \in F'_1$ such that $f'_1(x') = \psi_0(x) - h_{-1}f_0(x)$ and define $h_1(x) = x'$. This is possible, because $f'_0(\psi_0(x) - h_{-1}f_0)(x) = (\psi_{-1}f_0 - h_{-1}f_0)(x) = h_{-2}f_{-1}f_0(x) = 0$. Inductively, we obtain the chain homotopy h. ☐

LEMMA 3.9. *Let $F_\bullet$ and $F'_\bullet$ be free resolutions of H. Then there are canonical isomorphism*

$$H_n(F_\bullet \otimes G) \xrightarrow{\sim} H_n(F'_\bullet \otimes G)$$

induced by a chain homotopy equivalence $F_\bullet \longrightarrow F'_\bullet$.

Proof. By lemma 3.8 we may extend $\mathrm{id}\colon H \longrightarrow H$ to chain maps $F_\bullet \longrightarrow F'_\bullet$ and $F'_\bullet \longrightarrow F_\bullet$. These maps are chain homotopy equivalences because such extensions are unique up to chain homotopy and of course remain chain homotopy equivalences upon applying any additive functor. Hence, after tensoring with G, they induce inverse maps on homology. ☐

Remark 3.10. Lemma 3.8 also implies immediately that $\mathrm{Tor}(\_,\_)\colon \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$ is a covariant functor in both arguments. The action on morphisms is given by extending over given free resolutions and taking homology and does not depend on the choice of resolution.

This lemma implies that our definition of $\mathrm{Tor}_n(H, G)$ does not in fact depend on the choice of free resolution of H. Because

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C_\bullet) \longrightarrow 0$$

is a free resolution of $H_n(C_\bullet)$, we see that $\ker(i_n \otimes \mathrm{id})$ in

$$B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes G \longrightarrow H_n(C_\bullet) \otimes G \longrightarrow 0$$

is precisely $\mathrm{Tor}_1(H_n(C_\bullet), G)$. We have thus proven the following theorem.

Theorem 3.11 (Universal coefficient theorem for homology). *If $C_\bullet$ is a chain complex of free abelian groups and G an abelian group, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C_\bullet) \otimes G \longrightarrow H_n(C_\bullet \otimes G) \longrightarrow \mathrm{Tor}(H_{n-1}(C_\bullet), G) \longrightarrow 0$$

which are split non–canonically.

Proof. It only remains to construct the splittings. The sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

is split exact, i. e. the inclusion $Z_n \longrightarrow C_n$ admits a retraction $r_n\colon C_n \longrightarrow Z_n$. Hence, by precomposition with r_n the quotient map $q_n\colon Z_n \longrightarrow H_n(C_\bullet)$ extends to C_n:

$$\begin{array}{ccc} C_n & \xrightarrow{\varphi_n} & H_n(C_\bullet) \\ \uparrow & \nearrow_{q_n} & \\ Z_n & & \end{array}$$

Therefore, we obtain a chain map $\varphi_\bullet\colon C_\bullet \longrightarrow H_\bullet(C_\bullet)$ where the homology $H_\bullet(C_\bullet)$ is considered as a chain complex with $H_n(C_\bullet)$ in degree n and with trivial boundary operator, and there is a chain map $\varphi_\bullet \otimes \mathrm{id}_G\colon C_\bullet \otimes G \longrightarrow H_\bullet(C_\bullet) \otimes G$. Then the induced map $\varphi_*\colon H_n(C_\bullet \otimes G) \longrightarrow H_n(C_\bullet) \otimes G$ on homology is the required splitting for all n. To see that this is the case, observe that by construction we have $\varphi_n \otimes \mathrm{id}_G|_{Z_n \otimes G} = q_n \otimes \mathrm{id}_G$. Passing to homology this map induces the identity. $\square$

Corollary 3.12. *For any pair (X, A) there are natural short exact sequences*

$$0 \longrightarrow H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \mathrm{Tor}(H_{n-1}(X, A), G) \longrightarrow 0$$

which split non–canonically.

Proposition 3.13. *The* Tor*–functor has the following properties:*

(i) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.
(ii) $\mathrm{Tor}(\bigoplus_{i\in I} A_i, B) \cong \bigoplus_{i\in I} \mathrm{Tor}(A_i, B)$.
(iii) $\mathrm{Tor}(A, B) = 0$ *if* A *or* B *is torsionfree.*
(iv) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_{\mathrm{tor}}, B)$ *where* A_{tor} *denotes the torsion subgroup of* A.
(v) $\mathrm{Tor}(\mathbb{Z}/n, A) \cong \ker(A \xrightarrow{n} A)$.

(vi) *If*

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

is exact, then there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A, C) \longrightarrow \mathrm{Tor}(A, D) \longrightarrow A \otimes B \longrightarrow A \otimes C \longrightarrow A \otimes D \longrightarrow 0.$$

Proof.

(i) Apply (vi) to a free resolution $0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$ of B to obtain a long exact sequence

$$0 \longrightarrow \mathrm{Tor}(A, B) \longrightarrow A \otimes F_1 \longrightarrow A \otimes F_0 \longrightarrow A \otimes B \longrightarrow 0$$

where the terms to the right vanish because $\mathrm{Tor}(X, Y) = 0$ if X or Y is free. On the other by definition there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}(B, A) \longrightarrow F_1 \otimes A \longrightarrow F_0 \otimes A \longrightarrow B \otimes A \longrightarrow 0$$

computing $\mathrm{Tor}(B, A)$. But the tensor product is naturally symmetric and the five lemma gives a natural isomorphism $\mathrm{Tor}(A, B) \xrightarrow{\sim} \mathrm{Tor}(B, A)$.

(ii) Choose a direct sum of free resolutions of the A_i as a free resolution of $\bigoplus_{i \in I} A_i$.

(iii) If A or B is free, then of course $\mathrm{Tor}(A, B) = 0$. By (i) it is enough to check that for a free resolution

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow A \longrightarrow 0$$

and B torsion free the map $F_1 \otimes B \longrightarrow F_0 \otimes B$ is injective. Take $\sum x_i \otimes b_i \in F_1 \otimes B$ such that $(f_1 \otimes \mathrm{id})(\sum x_i \otimes b_i) = \sum f(x_i) \otimes b_i = 0$. Let $B' = (b_1, \ldots, b_k) \subset B$ be the subgroup generated by $b_1, \ldots, b_k$. Then B' is finitely generated and torsion free whence free. By the free case we have $\sum x_i \otimes b_i = 0 \in F_1 \otimes B'$ which already implies $\sum x_i \otimes b_i = 0 \in F_1 \otimes B$.

(iv) Apply (vi) to the short exact sequence

$$0 \longrightarrow A_{\mathrm{tor}} \longrightarrow A \longrightarrow A/A_{\mathrm{tor}} \longrightarrow 0$$

to obtain an exact sequence

$$0 \longrightarrow \mathrm{Tor}(A_{\mathrm{tor}}, B) \longrightarrow \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A/A_{\mathrm{tor}}, B).$$

But A/A_{tor} is torsion free and hence there is an isomorphism $\mathrm{Tor}(A_{\mathrm{tor}}, B) \cong \mathrm{Tor}(A, B)$ by (iii).

(v) Tensor the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

of $\mathbb{Z}/n$ with A to obtain the chain complex

$$0 \longrightarrow A \xrightarrow{n} A \longrightarrow A \otimes \mathbb{Z}/n \longrightarrow 0$$

whose homology computes Tor.

(vi) Truncate a free resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

of A to $0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$ and tensor with the given exact sequence:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes B & \longrightarrow & F_1 \otimes C & \longrightarrow & F_1 \otimes D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This is an exact sequence of chain complexes and induces the required long exact sequence.

Remark 3.14. Proposition 3.13 implies that $\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/d \cong \mathbb{Z}/n \otimes \mathbb{Z}/m$ with $(d) = (n, m)$. Furthermore, $\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) = \mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}) = 0$. Hence, for finitely generated abelian groups A and B there is an isomorphism $\mathrm{Tor}(A, B) \cong A_{\mathrm{tor}} \otimes B_{\mathrm{tor}}$. This justifies the term "torsion product" for $\mathrm{Tor}(A, B)$.

3.2 The Universal Coefficient Theorem for Cohomology

Consider a chain complex $C_\bullet$ of free abelian groups and let G be an arbitrary abelian group. Similarly to the previous section, we will now try to compute $H^\bullet(\mathrm{Hom}(C_\bullet, G))$ from $H_\bullet(C_\bullet)$ and G. The proof of the following lemma is an easy exercise in diagram chasing.

Lemma 3.15. *The functor* $\mathrm{Hom}(\_, G)$ *is left exact, i. e. any exact sequence*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G).$$

Since by lemma 3.2 additive functors always preserve split exact sequences, in particular the functor $\mathrm{Hom}(\_, G)\colon \mathbf{Ab} \longrightarrow \mathbf{Ab}$ does. We will proceed analogously to the homological case: Consider the short exact sequence

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

of chain complexes where again $Z_\bullet$ and $B_\bullet$ carry the trivial differential. This sequence splits termwise, hence

$$0 \longrightarrow \mathrm{Hom}(B_{\bullet-1}, G) \longrightarrow \mathrm{Hom}(C_\bullet, G) \longrightarrow \mathrm{Hom}(Z_\bullet, G) \longrightarrow 0$$

is again exact. We obtain the long exact sequence

$$\ldots \longrightarrow \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{\delta_{n-1}} \mathrm{Hom}(B_{n-1}, G) \longrightarrow H^n(\mathrm{Hom}(C_\bullet, G)) \longrightarrow \mathrm{Hom}(Z_{n,G}) \xrightarrow{\delta_n} \ldots$$

in cohomology where δ_n is just given by the restriction map $i_n^*\colon \mathrm{Hom}(Z_n, G) \longrightarrow \mathrm{Hom}(B_n, G)$; here $i_n\colon B_n \longrightarrow Z_n$ denotes the inclusion. This long exact sequence gives rise to short exact sequences

$$0 \longrightarrow \mathrm{coker}(i_{n-1}^*) \longrightarrow H^n(\mathrm{Hom}(C_\bullet, G)) \longrightarrow \ker(i_n^*) \longrightarrow 0$$

for every $n \in \mathbb{Z}$. Now it is clear that

$$\ker i_n^* = \{\varphi \in \mathrm{Hom}(Z_n, G) : \varphi|_{B_n} = 0\}$$

or, equivalently, $\ker(i_n^*)$ consists of all homomorphisms $\varphi\colon Z_n \longrightarrow G$ that factor through the homology $H_n(C_\bullet)$. Hence, $\ker(i_n^*) = \mathrm{Hom}(H_n(C_\bullet), G)$. To determine $\mathrm{coker}(i_n^*)$ consider the free resolution

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C_\bullet) \longrightarrow 0$$

of $H_n(C_\bullet)$. Applying $\mathrm{Hom}(\_, G)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow \mathrm{Hom}(Z_n, G) \xrightarrow{i_n^*} \mathrm{Hom}(B_n, G) \longrightarrow \mathrm{coker}(i_n^*) \longrightarrow 0.$$

Definition 3.16. For abelian groups H and G we define

$$\mathrm{Ext}^n(H, G) = H^n(\mathrm{Hom}(F_\bullet, G))$$

for a free resolution $F_\bullet$ of H.

Just as in the case of Tor this is well–defined up to canonical isomorphism and defines a contravariant functor $\mathrm{Ext}^n(\_, G)$ and a covariant functor $\mathrm{Ext}^n(H, \_)$. It is again clear that $\mathrm{Ext}^n(H, G) = 0$ for $n \geqslant 2$ and $n \leqslant 0$. Because of this vanishing we also write $\mathrm{Ext}(H, G) = \mathrm{Ext}^1(H, G)$. The proof of the following theorem is now essentially the same as in the homological case.

Theorem 3.17 (Universal coefficient theorem for cohomology). *Let $C_\bullet$ be a chain complex of free abelian groups and G an arbitrary abelian group. Then there are natural short exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(\mathrm{Hom}(C_\bullet, G)) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

which split non–canonically.

Corollary 3.18. *For any pair (X, A) of topological spaces, there are natural short exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X, A), G) \longrightarrow H^n(X, A; G) \longrightarrow \mathrm{Hom}(H_n(X, A), G) \longrightarrow 0$$

which split non–canonically.

Proposition 3.19. *For abelian groups A, B and A_i, $i \in I$, one has:*

(i) $\mathrm{Ext}(\bigoplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Ext}(A_i, B)$.
(ii) $\mathrm{Ext}(A, B) = 0$ *if A is free.*
(iii) $\mathrm{Ext}(\mathbb{Z}/n, B) \cong B/nB$.

There also is a six term exact sequence for Ext *analogous to the case of* Tor.

Proof.

(i) Choose a direct sum of free resolutions of the A_i as a free resolution of $\bigoplus_{i\in I} A_i$; therefore we have an exact sequence

$$0 \longrightarrow \bigoplus_{i\in I} F_{1,i} \longrightarrow \bigoplus_{i\in I} F_{0,i} \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow 0.$$

Applying $\mathrm{Hom}(\_, B)$ yields a cochain complex

$$0 \longrightarrow \prod_{i\in I} \mathrm{Hom}(A_i, B) \longrightarrow \prod_{i\in I} \mathrm{Hom}(F_{0,i}, B) \longrightarrow \prod_{i\in I} \mathrm{Hom}(F_{1,i}, B) \longrightarrow 0$$

whose only non–trivial homology in degree 1 is isomorphic to $\prod_{i\in I} \mathrm{Ext}(A_i, B)$ and computes $\mathrm{Ext}(\bigoplus_{i\in I} A_i, B)$.

(ii) If A is free, then $0 \longrightarrow A \longrightarrow A \longrightarrow 0$ is a free resolution of A, whence $\mathrm{Ext}(A, B) = 0$ for all B.

(iii) Apply $\mathrm{Hom}(\_, B)$ to the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

of $\mathbb{Z}/n$ to obtain a cochain complex

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/n, B) \longrightarrow \mathrm{Hom}(\mathbb{Z}, B) \xrightarrow{n} \mathrm{Hom}(\mathbb{Z}, B) \longrightarrow 0$$

whose homology computes $\mathrm{Ext}^\bullet(\mathbb{Z}/n, B)$. Concretely, $\mathrm{Ext}(\mathbb{Z}/n, B)$ is isomorphic to the cokernel of $B \xrightarrow{n} B$,i. e. to B/nB. □

Corollary 3.20. *If A is finitely generated, then* $\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(A_{\mathrm{tor}}, \mathbb{Z}) \cong A_{\mathrm{tor}}$.

Corollary 3.21. *If $H_n(X, A)$ and $H_{n-1}(X, A)$ are finitely generated, then there is a non–canonical decomposition*

$$H^n(X, A; \mathbb{Z}) \cong \mathrm{Hom}(H_n(X, A), \mathbb{Z}) \oplus H_{n-1}(X, A)_{\mathrm{tor}} \cong \frac{H_n(X, A)}{H_n(X, A)_{\mathrm{tor}}} \oplus H_{n-1}(X, A)_{\mathrm{tor}}.$$

4 The Axiomatic Point of View

The goal of this section is to sketch how the functor $H_n\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$ and $H^n\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$ are characterised by a set of axioms, known as the *Eilenberg–Steenrod axioms.* We will use the restriction functor $\mathrm{res}\colon \mathbf{Top}^2 \longrightarrow \mathbf{Top}^2$ given on objects by $\mathrm{res}(X, A) = (A, \emptyset)$.

Definition 4.1. A *homology theory* $h_\bullet$ consists of a sequence of functors $h_n\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$, $n \in \mathbb{Z}$, together with natural transformations $\partial_n\colon h_n \longrightarrow (h_{n-1} \circ \mathrm{res})$ satisfying the following axioms:

(i) If $f, g\colon (X, A) \longrightarrow (Y, B)$ are homotopic as maps of pairs, then $h_n(f) = h_n(g)$ for all $n \in \mathbb{Z}$. This is called *homotopy invariance.*

(ii) There is a long exact sequence

$$\ldots \longrightarrow h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A, \emptyset) \longrightarrow \ldots$$

induced by the inclusions $i\colon (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $j\colon (X, \emptyset) \longrightarrow (X, A)$.

(iii) A version of *excision* is satisfied: For any pair (X, A) with $\overline{B} \subset A^\circ$ the canonical inclusion map $i\colon (X \smallsetminus B, A \smallsetminus B) \hookrightarrow (X, A)$ induces isomorphisms

$$h_n(i)\colon h_n(X \smallsetminus B, X \smallsetminus A) \xrightarrow{\sim} h_n(X, A)$$

for all $n \in \mathbb{Z}$.

A *cohomology theory* $h^\bullet$ consists of a sequence of functors $h^n\colon \mathbf{Top}^2 \longrightarrow \mathbf{Ab}$, $n \in \mathbb{Z}$, together with natural transformations $\delta^n\colon h^n \longrightarrow (h^{n-1} \circ \mathrm{res})$ satisfying obvious dual versions of the Eilenberg–Steenrod axioms.

Given a homology theory $h_\bullet$ we write $h_n(X) = h_n(X, \emptyset)$ and use f_* instead of $h_n(f)$ for simplicity. The sequence of groups $h_n(*)$, $n \in \mathbb{Z}$ is called the *coefficients* of $h_\bullet$. Similar definitions are made for cohomology theories.

Definition 4.2. A homology theory $h_\bullet$ is called *ordinary* if $h_n(*) = 0$ for $n \neq 0$ and similarly for cohomology.

Theorem 4.3. *Singular (co–)homology is an ordinary (co–)homology theory.*

Theorem 4.4. *The values of an ordinary (co–)homology theory on the category of CW–complexes are determined by the coefficients.*

Proof. The construction of the cellular (co–)chain complex and the proof of the isomorphism of cellular with singular (co–)homology only used the Eilenberg–Steenrod axioms. □

Lemma 4.5. *If* $(X_1, A_1), \ldots, (X_k, A_k)$ *are pairs of topological spaces, then the map*

$$\bigoplus_{i=1}^{n} h_n(X_i, A_i) \longrightarrow h_n(X, A), \quad \textit{where } X = \coprod_{i=1}^{k} X_i \textit{ and } A = \coprod_{i=1}^{k} A_i,$$

induced by the inclusions $\iota_i\colon (X_i, A_i) \hookrightarrow (X, A)$ *is an isomorphism for all* $n \in \mathbb{Z}$. *A dual version holds for cohomology.*

Proof. Induction immediately reduces to the case $k = 2$ and the 5–lemma implies that it is enough to consider the case $A_1 = A_2 = \emptyset$. There is a commutative diagram

$$\begin{array}{ccc} h_n(X_1) & \xrightarrow{\quad\cong\quad} & h_n(X, X_2) \\ & \searrow \qquad \nearrow & \\ & h_n(X) & \\ & \nearrow \qquad \searrow & \\ h_n(X_2) & \xrightarrow[\quad\cong\quad]{} & h_n(X, X_1) \end{array}$$

where all maps are induced by the inclusions and the horizontal maps are isomorphisms by excision. The long exact sequence implies that the two diagonal sequences are exact. This is enough to imply the lemma. This proof obviously dualises to give the version for cohomology. ☐

Definition 4.6. A homology theory $h_\bullet$ is called *additive*, if for any pairs family (X_i, A_i), $i \in I$, of pairs, indexed by an *arbitrary* set I, the map

$$\bigoplus_{i \in I} h_n(X_i, A_i) \longrightarrow h_n(X, A), \quad \text{where } X = \coprod_{i \in I} X_i \text{ and } A = \coprod_{i \in I} A_i,$$

induced by the inclusions $\iota_i \colon (X_i, A_i) \hookrightarrow (X, A)$, is an isomorphism for all $n \in \mathbb{Z}$. Dually, a cohomology theory is called *additive* if it satisfies the obvious dual version of this statement.

Proposition 4.7. *Singular homology and cohomology are additive.*

For any homology theory $h_\bullet$ we define the *reduced* theory $\widetilde{h}_n(X) = \ker(h_n(X) \longrightarrow h_n(*))$. Dually, for any cohomology theory $h^\bullet$ we define $\widetilde{h}^n(X) = \operatorname{coker}(h^n(X) \longrightarrow h^n(*))$. Any choice of a base point $x_0 \colon * \longrightarrow X$ induces splittings of the exact sequences

$$0 \longrightarrow \widetilde{h}_n(X) \longrightarrow h_n(X) \longrightarrow h_n(*) \longrightarrow 0$$

and

$$0 \longrightarrow h^n(*) \longrightarrow h^n(X) \longrightarrow \widetilde{h}^n(X) \longrightarrow 0.$$

A map $f \colon X \longrightarrow Y$ clearly defines induced maps $f_* \colon \widetilde{h}_n(X) \longrightarrow \widetilde{h}_n(Y)$ and $f^* \colon \widetilde{h}^n(Y) \longrightarrow \widetilde{h}^n(X)$ which make $\widetilde{h}_n$ and $\widetilde{h}^n$ into functors **Top** $\longrightarrow$ **Ab**. There is an axiomatic characterisation of reduced homology theories $\widetilde{h}_\bullet$ and reduced cohomology theories $\widetilde{h}^\bullet$ as above, e. g. there is always a long exact sequence

$$\ldots \longrightarrow \widetilde{h}_n(A) \longrightarrow \widetilde{h}_n(X) \longrightarrow h_n(X, A) \longrightarrow \widetilde{h}_{n-1}(X, A) \longrightarrow \ldots$$

in reduced homology.

5 Products and Duality

5.1 Cup Products

Let R be a commutative ring and consider singular cohomology $H^\bullet(X;R)$ with coefficients in R. The goal of this section is to construct the structure of a graded ring on $H^\bullet(X;R)$.

Definition 5.1. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ the cup product $\varphi \cup \psi \in C^{k+\ell}(X;R)$ is defined via

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma \circ [e_0, \dots, e_k]) \cdot \psi(\sigma \circ [e_k, \dots, e_{k+\ell}])$$

for all singular simplices $\sigma \colon \Delta^{k+\ell} \longrightarrow X$.

Lemma 5.2. *For* $\varphi \in C^k(X;R)$ *and* $\psi \in C^\ell(X;R)$ *we have the* graded Leibniz rule

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi.$$

Proof. For $\sigma \colon \Delta^{k+\ell+1} \longrightarrow X$ compute

$$\begin{aligned}(\delta\varphi \cup \psi)(\sigma) &= (\delta\varphi)(\sigma \circ [e_0, \dots, e_{k+1}]) \cdot \psi(\sigma \circ [e_{k+1}, \dots, e_{k+\ell+1}]) = \\ &= \sum_{i=0}^{k+1} (-1)^i \; \varphi(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{k+1}]) \cdot \psi(\sigma \circ [e_{k+1}, \dots, e_{k+\ell+1}])\end{aligned}$$

and

$$\begin{aligned}(-1)^k(\varphi \cup \delta\psi)(\sigma) &= (-1)^k \; \varphi(\sigma \circ [e_0, \dots, e_k]) \cdot (\delta\psi)(\sigma \circ [e_k, \dots, e_{k+\ell+1}]) = \\ &= \sum_{i=k}^{k+\ell+1} (-1)^k \; (-1)^{i-k} \; \varphi(\sigma \circ [e_0, \dots, e_k]) \cdot \psi(\sigma \circ [e_k, \dots, \widehat{e_i}, \dots, e_{k+\ell+1}]).\end{aligned}$$

Combining, we see that

$$(\delta\varphi \cup \psi)(\sigma) + (-1)^k(\varphi \cup \delta\psi)(\sigma) = \sum_{i=0}^{k+\ell+1} (-1)^i \; (\partial_i(\varphi \cup \psi))(\sigma) = (\delta(\varphi \cup \psi))(\sigma).$$

□

Corollary 5.3. *The cup product descends to a pairing*

$$\cup \colon H^k(X;R) \otimes H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R),$$

the cup product *in singular cohomology. It is associative, distributive and unital if* R *is.*

Definition 5.4. The abelian group

$$H^*(X;R) = \bigoplus_{n \in \mathbb{Z}} H^n(X;R)$$

becomes a graded ring with the cup product, the *singular cohomology ring* of X with coefficients in R.

There are relative versions of the cup product

$$H^k(X;R) \otimes H^\ell(X,A;R) \xrightarrow{\cup} H^{k+\ell}(X,A;R), \quad H^k(X,A;R) \otimes H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X,A;R)$$

and

$$H^k(X,A;R) \otimes H^\ell(X,A;R) \xrightarrow{\cup} H^{k+\ell}(X,A;R)$$

since if φ or ψ vanishes on $C_\bullet(A)$, then so does $\varphi \cup \psi$.

Lemma 5.5. *For a map* $f\colon X \longrightarrow Y$ *the induced map* $f^*\colon H^*(X;R) \longrightarrow H^*(X;R)$ *is a homomorphism of graded rings, i. e.*

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$$

for $\alpha, \beta \in H^*(X;R)$ *and* $\deg(f^*\alpha) = \deg\alpha$.

Proof. It is enough to check this for homogeneous elements α and β. We may compute

$$\begin{aligned}\big(f^*\varphi \cup f^*\psi\big) &= f^*\varphi(\sigma \circ [e_0, \dots, e_k]) \cdot f^*\psi(\sigma \circ [e_k, \dots, e_{k+\ell}])\\ &= (\varphi \cup \psi)(f \circ \sigma) = \big(f^*(\varphi \cup \psi)\big)(\sigma)\end{aligned}$$

for cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ and every simplex $\sigma\colon \Delta^{k+\ell} \longrightarrow Y$. The result follows. □

Proposition 5.6. *The cup product is graded commutative, i. e.* $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ *for homogeneous elements* $\alpha, \beta \in H^*(X, A; R)$.

Proof. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ we have

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma \circ [e_0, \dots, e_k]) \cdot \psi(\sigma \circ [e_k, \dots, e_{k+\ell}])$$

and

$$(\psi \cup \varphi)(\sigma) = \varphi(\sigma \circ [e_\ell, \dots, e_{k+\ell}]) \cdot \psi(\sigma \circ [e_0, \dots, e_\ell]).$$

Consider the affine singular simplex

$$\tau_n = [e_n, \dots, e_0]\colon \Delta^n \longrightarrow \Delta^n$$

and define a homomorphism $\rho_n\colon C_n(X) \longrightarrow C_n(X)$ via $\rho_n(\sigma) = \varepsilon_n\, \sigma \circ \tau_n$ with $\varepsilon_n = (-1)^{n(n+1)/2}$. We claim that $\rho_\bullet\colon C_\bullet(X) \longrightarrow C_\bullet(X)$ is a chain map which is chain homotopic to the identity. Indeed,

$$\partial\rho_n(\sigma) = \varepsilon_n\, \partial(\sigma \circ \tau_n) = \varepsilon_n \sum_{i=0}^{n} (-1)^i\, \sigma \circ [e_n, \dots, \widehat{e_{n-i}}, \dots, e_0]$$

and

$$\begin{aligned}\rho_{n-1}\partial(\sigma) &= \rho_{n-1} \sum_{i=0}^{n} (-1)^i\, \sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_n] = \varepsilon_{n-1} \sum_{i=0}^{n} (-1)^i\, \sigma \circ [e_n, \dots, \widehat{e_i}, \dots, e_0] =\\ &= \varepsilon_{n-1} \sum_{i=0}^{n} (-1)^{n-i}\, \sigma \circ [e_n, \dots, \widehat{e_{n-i}}, \dots, e_0] = \varepsilon_n \sum_{i=0}^{n} (-1)^i\, \sigma \circ [e_n, \dots, \widehat{e_{n-i}}, \dots, e_0].\end{aligned}$$

In the product $\Delta^n \times I \subset \mathbb{R}^{n+1} \times I$ identify e_i with $(e_i, 0)$ and f_i with $(e_i, 1)$. We define a homomorphism $h_n\colon C_n(X) \longrightarrow C_{n+1}(X)$ by the formula

$$h_n(\sigma) = \sum_i (-1)^i \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \dots, e_i, f_n, \dots, f_i]$$

with the projection $\pi\colon \Delta^n \times I \longrightarrow \Delta^n$. It corresponds to the usual triangulation of $\Delta^n \times I$ with some orientations reversed. Then

$$\begin{aligned}\partial h_n(\sigma) = &\sum_{i \leqslant j} (-1)^j (-1)^i \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \dots, \widehat{e_j}, \dots, e_i, f_n, \dots, f_i] +\\ &+ \sum_{j \geqslant i} (-1)^{i+1+n-j} (-1)^i \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \dots, e_i, f_n, \dots, \widehat{f_j}, \dots, f_i].\end{aligned}$$

The $j = i$ terms in this sum give

$$\varepsilon_n \sigma \circ \pi \circ [f_n, \ldots, f_0] + \sum_{i>0} \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \ldots, e_{i-1}, f_n, \ldots, f_i] + \\ + \sum_{i<n} (-1)^i (-1)^{1+n} \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \ldots, e_i, f_n, \ldots, f_{i+1}] - \sigma \circ \pi \circ [e_0, \ldots, e_n].$$

The outer two term in this sum are

$$\varepsilon_n \sigma \circ [e_n, \ldots, e_0] - \sigma \circ [e_0, \ldots, e_n] = \rho_n(\sigma) - \sigma$$

and an index shift in the second sum in the middle terms reveals that the middle terms cancel. Now,

$$\begin{aligned} h_{n-1}(\partial\sigma) &= \sum_j (-1)^j \sum_{i \leqslant n-1} (-1)^i e_{n-1-i} \sigma \circ [e_0, \ldots, \widehat{e_j}, \ldots, e_n] \circ \pi \circ [e_0, \ldots, e_i, f_n, \ldots, f_i] = \\ &= \sum_{i<j} (-1)^j (-1)^i \varepsilon_{n-1-i} \sigma \circ \pi \circ [e_0, \ldots, e_i, f_n, \ldots, \widehat{f_j}, \ldots, f_i] + \\ &\quad + \sum_{i>j} (-1)^j (-1)^{i-1} \varepsilon_{n-i} \sigma \circ \pi \circ [e_0, \ldots, \widehat{e_j}, \ldots, e_i, f_n, \ldots, f_i] \end{aligned}$$

and comparing with the expression for $\partial h_n(\sigma)$ shows that $h_\bullet$ is a chain homotopy from $\rho_\bullet$ to id. To finish the proof, observe that

$$\begin{aligned} (\rho_k^* \varphi \cup \rho_\ell^* \psi)(\sigma) &= (\rho_k^* \varphi)(\sigma \circ [e_0, \ldots, e_k]) \cdot (\rho_\ell^* \psi)(\sigma \circ [e_k, \ldots, e_{k+\ell}]) = \\ &= \varepsilon_k \varepsilon_\ell \varphi(\sigma \circ [e_k, \ldots, e_0]) \cdot \psi(\sigma \circ [e_{k+\ell}, \ldots, e_k]) \end{aligned}$$

and

$$\rho_{k+\ell}^* (\psi \cup \varphi)(\sigma) = \varepsilon_{k+\ell} \psi(\sigma \circ [e_{k+\ell}, \ldots, e_k]) \cdot \varphi(\sigma \circ [e_k, \ldots, e_0]).$$

Hence, on cohomology this implies that $[\varphi] \cup [\psi] = \varepsilon_k \varepsilon_\ell \varepsilon_{k+\ell} [\psi \cup \varphi] = (-1)^{k\ell} [\psi] \cup [\varphi]$. Since ρ_n and h_n take chains in A to chains in A, they act on relative cochains and the same argument works for $H^*(X, A; R)$. ☐

Example 5.7. Consider the 2–torus T^2 given by its fundamental polygon with vertices A, B, C and D—see Bildchen. The Hurewicz theorem implies that $a = [A, B]$ and $b = [A, D]$ give a basis for $H_1(T^2)$. The homology $H_\bullet(T^2; \mathbb{Z})$ is torsionfree and therefore $H^1(T^2; \mathbb{Z}) = \mathrm{Hom}(H_1(T^2), \mathbb{Z})$. Let $\{\alpha, \beta\}$ be the basis of $H^1(T^2; \mathbb{Z})$ dual to $\{a, b\}$. Write $c = [A, B, D] - [B, D, C]$. This singular 2–cycle generates $H_2(T^2)$. For $\xi = [\varphi] \in H^1(T^2; \mathbb{Z})$ and $\eta = [\psi] \in H^1(T^2; \mathbb{Z})$ we have

$$(\xi \cup \eta)([c]) = \varphi([A, B])\psi([B, D]) - \varphi([B, D])\psi([D, C]) = \xi(a)\eta(b - a) - \xi(b - a)\eta(a)$$

because $[B, D] - (b - a) = \partial[A, B, D] \equiv 0 \in H_1(T^2)$. Hence,

$$(\xi \cup \eta)([c]) = \xi(a)\eta(b) - \xi(b)\eta(a)$$

and in particular $\alpha \cup \alpha = 0$, $\beta \cup \beta = 0$ and $(\alpha \cup \beta)([c]) = 1$. Therefore $\alpha \cup \beta$ generates $H^2(T^2; \mathbb{Z})$ and $[c]$ generates $H_2(T^2)$. In summary, we have calculated $H^\bullet(T^2; \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}} \mathbb{Z}^2 = \bigwedge_{\mathbb{Z}} \langle \alpha, \beta \rangle$.

Example 5.8. More generally, let Σ_g be the oriented surface of genus g, presented by its fundamental polygon—see Bildchen. As before, the homology classes $a_1, b_1, \dots, a_g, b_g$ of the singular 1–cycles $[A_1, B_1], [B_1, C_1], \dots, [A_g, B_g], [B_g, C_g]$ generate the first homology $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. In fact, they form a basis. Let $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ be the corresponding dual basis. Write

$$C = \sum_{i+1 \in \mathbb{Z}/g} [0, A_i, B_i] + [0, B_i, C_i] - [0, D_i, C_i] - [0, A_{i+1}, D_i].$$

Then

$$\partial C = \sum_{i+1 \in \mathbb{Z}/g} [0, A_i] - [0, A_{i+1}] = 0.$$

As before,

$$\begin{aligned}(\xi \cup \eta)([c]) &= \sum_{i+1 \in \mathbb{Z}/g} \varphi([0, A_i])\psi(a_i) + \varphi([0, B_i])\psi(b_i) - \varphi([0, D_i])\psi(a_i) - \varphi([0, A_{i+1}])\psi(b_i) = \\ &= \sum_{i+1 \in \mathbb{Z}/g} \varphi([0, A_i] - [0, D_i])\eta(a_i) + \varphi([0, B_i] - [0, A_{i+1}])\eta(b_i).\end{aligned}$$

Now, we have

$$\partial([0, A_i, B_i] + [0, B_i, C_i] + [0, C_i, D_i]) - [B_i, C_i] = [0, A_i] - [0, D_i]$$

and

$$\partial([0, B_i, C_i] + [0, C_i, D_i] + [0, D_i, A_{i+1}]) + [A_i, B_i] = [0, B_i] - [0, A_{i+1}]$$

and therefore

$$(\xi \cup \eta)([c]) = \sum_{i+1 \in \mathbb{Z}/g} \xi(a_i)\eta(b_i) - \xi(b_i)\eta(a_i).$$

We obtain the relations $\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$ and $(\alpha_i \cup \beta_j)([c]) = \delta_{ij}$. In particular, the cohomology class $\alpha_1 \cup \beta_1 = \dots = \alpha_g \cup \beta_g$ generates $H^2(\Sigma_g; \mathbb{Z})$ and $[c]$ generates $H_2(\Sigma_g)$.

Our next goal will be to generalise this computation to $H^\bullet(T^n; \mathbb{Z})$. Here $T^n = S^1 \times \dots \times S^1$ denotes the n–torus. First, we need some preparation.

Definition 5.9. The *cross product* is the map

$$\times\colon H^k(X; R) \otimes H^\ell(X; R) \longrightarrow H^{k+\ell}(X \times Y; R)$$

defined by $\alpha \times \beta = \pi_X^* \alpha \cup \pi_Y^* \beta$.

We would like to define a relative version of the cross product

$$\times\colon H^k(X, A; R) \otimes H^\ell(Y, B; R) \longrightarrow H^{k+\ell}(X \times Y, X \times B \cup A \times Y; R).$$

The naive version does not quite work: For cochains $\varphi \in C^k(X, A; R)$ and $\psi \in C^\ell(Y, B; R)$ we only have $\pi_X^* \varphi \cup \pi_Y^* \psi \in C^{k+\ell}(X \times Y, A \times Y + X \times B; R)$, i. e. the cup product lands in the space of all $\xi \in C^{k+\ell}(X \times Y; R)$ such that $\xi|_{C_\bullet(A \times Y)} = \xi|_{C_\bullet(X \times B)}$ which is bigger than $C^{k+\ell}(X \times Y, A \times Y \cup X \times B; R)$. So we need a more general version of the cup product. Ideally, it should be a pairing

$$\cup\colon H^k(X, A; R) \otimes H^\ell(X, B; R) \longrightarrow H^{k+\ell}(X, A \cup B; R).$$

For this we need to understand the subcomplexes $C_\bullet(A\cup B)$ and $C_\bullet(A)+C_\bullet(B)$ of $C_\bullet(X)$. Call $A, B \subset X$ an *excisive couple* if the inclusion $C_\bullet(A) + C_\bullet(B) \hookrightarrow C_\bullet(A \cup B)$ induces an isomorphism in homology. In this case, the universal coefficient theorem implies that also the dual $H^n(A\cup B; R) \longrightarrow H^n(A+B; R)$ is an isomorphism for all n. Here, $H^n(A + B; R)$ denotes $H^n \operatorname{Hom}(C_\bullet(A) + C_\bullet(B), R)$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\bullet(X, A+B; R) & \longrightarrow & C^*(X; R) & \longrightarrow & C^*(A+B; R) & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & C^\bullet(X, A\cup B; R) & \longrightarrow & C^*(X; R) & \longrightarrow & C^*(A\cup B; R) & \longrightarrow & 0 \end{array}$$

of cochain complexes with exact rows. We obtain a diagram

$$\begin{array}{ccccccccc} \longrightarrow & H^n(X, A+B; R) & \longrightarrow & H^n(X; R) & \longrightarrow & H^n(A+B; R) & \longrightarrow & H^{n+1}(X, A+B; R) & \longrightarrow \\ & \uparrow & & \| & & \uparrow & & \uparrow & \\ \longrightarrow & H^n(X, A\cup B; R) & \longrightarrow & H^n(X; R) & \longrightarrow & H^n(A\cup B; R) & \longrightarrow & H^{n+1}(X, A\cup B; R) & \longrightarrow \end{array}$$

with exact rows in which every map $H^n(A\cup B; R) \longrightarrow H^n(A+B; R)$ is an isomorphism by assumption. Therefore, the five lemma implies that $H^n(X, A\cup B; R) \longrightarrow H^n(X, A+B; R)$ is an isomorphism as well. This isomorphism now gives a definition of the relative cup product, at least for excisive couples; it is the composition

$$H^k(X, A; R) \otimes H^\ell(X, B; R) \xrightarrow{\cup} H^{k+\ell}(X, A+B; R) \xrightarrow{\sim} H^{k+\ell}(X, A\cup B; R).$$

Of course, we need to determine which couples are excisive. By the excision theorem, this is the case if $A\cup B = A^\circ \cup B^\circ$, where the interiors are understood with respect to the subspace topology on $A\cup B$. This holds for example if $A, B \subset X$ are both open.

Returning to our original setting, we have a *relative cross product*

$$H^k(X, A; R) \otimes H^\ell(Y, B; R) \longrightarrow H^{k+\ell}(X\times Y, A\times Y \cup X\times B; R)$$

if the subspaces $A\times Y$ and $X\times B$ of $X\times Y$ form an excisive couple. For instance, this is the case if $A\subset X$ and $B\subset Y$ are both open.

Our next goal will be computing the ring structure on $H^\bullet(T^n; \mathbb{Z})$, the cohomology of the n–torus $T^n = S^1\times\cdots\times S^1$.

LEMMA 5.10. *For any pair* (X, A) *and any space* Y *the diagram*

$$\begin{array}{ccc} H^k(A; R)\otimes H^\ell(Y; R) & \xrightarrow{\delta\otimes \mathrm{id}} & H^{k+1}(X, A; R)\otimes H^\ell(Y; R) \\ \downarrow\times & & \downarrow\times \\ H^{k+\ell}(A\times Y; R) & \xrightarrow{\delta} & H^{k+\ell+1}(X\times Y, A\times Y; R) \end{array}$$

is commutative.

Proof. Take $[\varphi]\otimes[\psi] \in H^k(A; R)\otimes H^\ell(Y; R)$. Then, we get a diagram

$$\begin{array}{ccc} [\varphi]\otimes[\psi] & \longmapsto & [\delta\overline{\varphi}]\otimes[\psi] \\ & & \downarrow \\ & & \pi_X^*[\delta\overline{\varphi}] \cup \pi_Y^*[\psi] \end{array}$$

where $\overline{\varphi}$ denotes some lift of φ to $C^k(X;R)$. The other way of chasing the diagram maps $[\varphi] \otimes [\psi]$ to $[\delta(\overline{\varphi} \times \psi)] = [\delta(\pi_X^* \overline{\varphi} \cup \pi_Y^* \psi)]$, and indeed by naturality of δ this equals $\pi_X^* \delta[\overline{\varphi}] \cup \pi_Y^*[\psi]$. □

Proposition 5.11. *Let* Y *be a topological space and* $I = [0,1]$ *the unit interval. Assume additionally that* R *is unital. If* α *is such that* $\langle \alpha \rangle_R = H^1(I, \partial I; R) \cong R$*, then the relative cross product*

$$H^n(Y;R) \xrightarrow{\alpha \times} H^{n+1}(I \times Y, \partial I \times Y; R)$$

with α *is an isomorphism.*

Proof. Consider the long exact sequence

$$\longrightarrow H^n(I \times Y, \partial I \times Y; R) \longrightarrow H^n(I \times Y; R) \longrightarrow H^n(\partial I \times Y; R) \longrightarrow H^{n+1}(I \times Y, \partial I \times Y; R) \longrightarrow$$

of $(I \times Y, \partial I \times Y)$ in cohomology. We claim that all maps $H^n(I \times Y, \partial I \times Y; R) \longrightarrow H^n(I \times Y; R)$ are 0. In fact, this map is induced by the inclusion $(I \times Y, \emptyset) \longrightarrow (I \times Y, \partial I \times Y)$. But this map is homotopic to a map with values in $\partial I \times Y$ which implies the claim.

Hence, we obtain short exact sequences

$$0 \longrightarrow H^n(I \times Y; R) \longrightarrow H^n(\partial I \times Y; R) \xrightarrow{\delta} H^{n+1}(I \times Y; \partial I \times Y; R) \longrightarrow 0$$

for all n. By additivity, $H^n(\partial I \times Y; R) \cong H^n(\{0\} \times Y; R) \oplus H^n(\{1\} \times Y; R)$. We now claim that for all $i \in \{0,1\}$ the restriction $\delta|_{H^n(\{i\} \times Y; R)}$ is an isomorphism: We can describe the submodule $H^n(\{i\} \times Y; R)$ of $H^n(\partial I \times Y; R)$ as follows. A basis of $H^0(\partial I; R)$ is given by cocyles 1_0 and 1_1 such that $1_i(j) = \delta_{ij}$ for $i, j \in \{0,1\}$. Hence, $H^n(\{i\} \times Y; R) = \mathrm{im}(1_i \times \_) \subset H^n(\partial I \times Y; R)$. On the other hand,

$$\mathrm{im}(H^n(I \times Y; R) \longrightarrow H^n(\partial I \times Y; R)) = \mathrm{im}(H^n(Y; R) \xrightarrow{\pi_Y^*} H^n(\partial I \times Y; R)).$$

Thus, the image of $H^n(I \times Y; R)$ in $H^n(\partial I \times Y; R)$ is generated by $(1_0 + 1_1) \times H^n(Y; R)$ and the $H^n(\{i\} \times Y; R)$ are complementary to $\ker \delta$. This yields the claim. To finish the proof, use lemma 5.10 to deduce that $\delta \circ (1_i \times \_) = \delta(1_i) \times \_$. The second δ denotes the connecting homomorphism $H^0(\partial I; R) \longrightarrow H^1(I, \partial I; R)$. Then observe that $\delta(1_0) = -\delta(1_1)$ is a generator of $H^1(I, \partial I; R)$ as before. □

Corollary 5.12. *If* α' *is a generator of* $H^1(S^1, \{s_0\}; R)$*, then*

$$H^n(Y; R) \xrightarrow{\alpha' \times} H^{n+1}(S^1 \times Y, \{s_0\} \times Y; R)$$

is an isomorphism for all n*.*

Corollary 5.13. *The homomorphism* $H^n(Y;R) \times H^{n+1}(Y;R) \longrightarrow H^{n+1}(S^1 \times Y; R)$ *mapping* (β_1, β_2) *to* $\alpha'' \times \beta_1 + 1_{H^0(S^1;R)} \times \beta_2$ *for some generator* α'' *of* $H^1(S^1; R)$ *is an isomorphism.*

Proposition 5.14. *There is an isomorphism* $H^\bullet(T^n; \mathbb{Z}) \cong \bigwedge \mathbb{Z}^n = \bigwedge_{\mathbb{Z}} \langle \alpha_1, \ldots, \alpha_n \rangle$ *where* $\alpha_i = \pi_i^* \alpha''$ *for a generator* α'' *of* $H^1(S^1; \mathbb{Z})$*.*

6 Orientations and Homology

Definition 6.1. A topological space X is called *locally Euclidean* if there is some $n \geqslant 0$ such that every $x \in X$ admits an open neighbourhood $U \subset X$ of x and a homeomorphism $\phi\colon U \longrightarrow V$ onto some open subset $V \subset \mathbb{R}^n$.

Definition 6.2. A *topological manifold* M is a locally Euclidean Hausdorff space. It is called *closed* if it is compact. If M is locally homeomorphic to $\mathbb{R}^n$, then $n \geqslant 0$ is called the *dimension* of M.

Example 6.3. The n–sphere S^n is a closed topological manifold by stereographic projection. Similarly, $\mathbb{RP}^n$, Σ_g and K^2 are examples of topological manifolds.

Let M be a topological manifold and fix $x \in M$. Choose some chart $\phi\colon U \longrightarrow V \subset \mathbb{R}^n$ centred at x, i. e. $\phi(x) = 0$. We may assume without loss of generality that $V = \mathbb{R}^n$. Then there are isomorphisms $H_k(M, M \setminus \{x\}; G) \cong H_k(U, U \setminus \{x\}; G) \cong \widetilde{H}_{k-1}(S^{n-1}; G) = \delta_{kn} G$. Hence, the dimension of M is detected by $H_*(M, M \setminus \{x\}; G)$.

Definition 6.4. Let R be a commutative ring with unity. If M is an n–dimensional topological manifold, then a generator $\mu_x \in H_n(M, M \setminus \{x\}; R) \cong R$ is called a *local orientation* at x.

Remark 6.5. In the case $R = \mathbb{Z}$ and $A \in O(n)$ we have $\deg A|_{S^{n-1}} = \det A$; in particular $\deg A|_{S^{n-1}} = 1$ if A is orientation preserving and $\deg A|_{S^{n-1}} = -1$ if A reverses orientation. Hence, for a local orientation $\mu \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \widetilde{H}_{n-1}(S^{n-1})$ we have $A_*\mu = \mu$ if A preserves orientation and $A_*\mu = -\mu$ otherwise.

For $x \in M$ and a chart $\phi\colon U \longrightarrow \mathbb{R}^n$ centred at x let $B^n = B_1^n(0)$ be the open unit ball and consider $B = \phi^{-1}(B^n) \subset U$.

Lemma 6.6. *The inclusion* $(M, M \setminus B) \longrightarrow (M, M \setminus \{x\})$ *of pairs induces a canonical isomorphism* $\rho_x\colon H_n(M, M \setminus B; R) \longrightarrow H_n(M, M \setminus \{x\}; R)$.

Proof. By excision, we have $H_n(M, M \setminus B; R) \cong H_n(U, U \setminus B; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n; R)$. On the other hand, $H_n(M, M \setminus \{x\}; R) \cong H_n(U, U \setminus \{x\}; R) \cong H_n(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}; R)$. By homotopy invariance, it follows that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B; R)$. ☐

This lemma implies that local orientations may be extended locally: Via the isomorphism ρ_x we get canonical identifications $H_n(M, M \setminus B; R) = H_n(M, M \setminus \{x\}; R)$ for all $x \in B$ and a local orientation μ_{x_0} at some $x_0 \in B$ extends to a map $\mu_B : B \longrightarrow \coprod_{x \in B} H_n(M, M \setminus \{x\}; R)$.

Remark 6.7. Let U' and U'' be open neighbourhoods of x_0 with open balls $B' \subset U'$ and $B'' \subset U''$ around x_0. Let $B \subset U$ be some ball around x_0 such that $B \subset B' \cap B''$. Then there is a commutative diagram

$$\begin{array}{ccccc} H_n(M, M \setminus B'; R) & & & \overset{\cong}{\searrow} & \\ & \searrow & & & \\ & & H_n(M, M \setminus B; R) & \xrightarrow{\cong} & H_n(M, M \setminus \{x_0\}; R) \\ & \nearrow & & & \\ H_n(M, M \setminus B''; R) & & & \underset{\cong}{\nearrow} & \end{array}$$

which implies that two leftmost maps are isomorphisms as well. This shows that $\mu_{B'}|_B = \mu_{B''}|_B = \mu_B$.

Let M_R be the disjoint unions $\coprod_{x\in M} H_n(M, M \smallsetminus \{x\}; R)$ and denote by $\pi\colon M_R \longrightarrow M$ the obvious projection. We define a topology on M_R as follows. If $\phi\colon U \longrightarrow \mathbb{R}^n$ is a chart and B is some ball in U, write

$$U_{\alpha_B} = \{\rho_x(\alpha_B) \in H_n(M, M \smallsetminus \{x\}; R) : x \in B\}$$

for every $\alpha_B \in H_n(M, M \smallsetminus B; R)$. Remark 6.7 shows that if $\rho_{x_0}(\alpha'_{B'}) = \rho_{x_0}(\alpha''_{B''})$ then $\alpha'_{B'}|_B = \alpha''_{B''}|_B$ for some ball $B \subset B' \cap B''$, i. e. the sets U_{α_B} form a basis for some topology on M_R. With this topology, the projection π becomes continuous and so do the local sections α_B as above. Moreover, the maps $B \times H_n(M, M \smallsetminus B; R) \longrightarrow \pi^{-1}(B)$ such that $(x, \alpha_B) \longmapsto \rho_x(\alpha_B)$ are local trivialisations of π. Therefore, we obtain a covering space $\pi\colon M_R \longrightarrow M$. This covering is called the R–*orientation bundle.*

Definition 6.8. A *(global)* R–*orientation* of M is a global section μ of $M_R \longrightarrow M$ such that every $\mu(x) \in H_n(M, M \smallsetminus \{x\}; R)$ is a local orientation around $x \in M$. If M admits a global R–orientation, we say it is R–*orientable.* In the case $R = \mathbb{Z}$ we simply speak of global orientations and orientable manifolds respectively.

Remark 6.9.

(i) If M is R–orientable, then $M_R \cong M \times R$ is trivial. In fact, if $\mu\colon M \longrightarrow M_R$ is a global R–orientation, then the map $M \times R \longrightarrow M_R$ mapping (x, r) to $r\mu(x)$ is a trivialisation of M_R.

(ii) Any topological manifold admits a unique $\mathbb{Z}/2$–orientation.

(iii) If a manifold M is not orientable, then the subspace

$$\widetilde{M} = \coprod_{x\in M} \{\mu_x \in H_n(M, M \smallsetminus \{x\}) : \mu_x \text{ is a generator}\} \subset M_{\mathbb{Z}}$$

is a nontrivial 2–sheeted covering of M—the *orientation covering.* The topological manifold $\widetilde{M}$ is canonically oriented: A local orientation $\widetilde{\mu}_x \in H_n(\widetilde{M}, \widetilde{M} \smallsetminus \{\mu_x\})$ is given by the element corresponding to μ_x via canonical the isomorphism $H_n(\widetilde{M}, \widetilde{M} \smallsetminus \{\mu_x\}) \cong H_n(M, M \smallsetminus \{x\})$.

Proposition 6.10. *If M is connected and $\pi_1(M)$ does not contain a subgroup of index 2, then M is orientable. In particular, this is the case if M is simply connected.*

Proof. This follows from remark 6.9 and the classification of covering spaces. ☐

Definition 6.11. A class $\mu_M \in H_n(M; R)$ is an R–*orientation class* if $\mu_x = \rho_x(\mu_M)$ is a local R–orientation for all $x \in M$. An *orientation class* is by definition a $\mathbb{Z}$–orientation class.

Remark 6.12. Clearly, if M admits an R–orientation class, then M is R–orientable.

We will study the converse of remark 6.12. For this we need a relative Mayer–Vietoris sequence. Assume that $X = A^\circ \cup B^\circ$ and $Y \subset X$ is a subspace such that $Y = C^\circ \cup D^\circ$ in the subspace topology and $C \subset A$, $D \subset B$. Then there is a long exact sequence

$$\ldots \longrightarrow H_n(A \cap B, C \cap D) \longrightarrow H_n(A, C) \oplus H_n(B, D) \longrightarrow H_n(X, Y) \longrightarrow H_{n-1}(A \cap B, C \cap D) \longrightarrow \ldots$$

induced by the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_\bullet(C\cap D) & \longrightarrow & C_\bullet(C)\oplus C_\bullet(D) & \longrightarrow & C_\bullet(C+D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_\bullet(A\cap B) & \longrightarrow & C_\bullet(A)\oplus C_\bullet(B) & \longrightarrow & C_\bullet(A+B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_\bullet(A\cap B, C\cap D) & \longrightarrow & C_\bullet(A,C)\oplus C_\bullet(B,D) & \longrightarrow & C_\bullet(A+B, C+D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all three rows are exact; note that the homology of $C_\bullet(A+B, C+D)$ computes $H_*(A\cup B, C\cup D)$ because of excision and our assumptions on the covers $X = A^\circ \cup B^\circ$ and $Y = C^\circ \cup D^\circ$.

Proposition 6.13. *Let* M *be a topological manifold of dimension* n *with a compact subset* $A \subset X$.

(i) *If* α *is a global section of* $\pi\colon M_R \longrightarrow M$*, then there exists a unique* $\alpha_A \in H_n(M, M \smallsetminus A; R)$ *such that* $\rho_x(\alpha_A) = \alpha(x)$ *for all* $x \in A$.

(ii) *For* $k > \dim M$ *the homology* $H_k(M, M \smallsetminus A; R)$ *vanishes.*

Proof. First we will show that if the proposition is true for A, B and $A \cap B$, then it is also true for $A \cup B$, where A and B are compact subsets of X. For this, we use the relative Mayer–Vietoris sequence for $M \smallsetminus (A \cup B) = (M \smallsetminus A) \cap (M \smallsetminus B)$ and $(M \smallsetminus A) \cup (M \smallsetminus B) = M \smallsetminus (A \cap B)$ to obtain

$$\ldots \longrightarrow H_k^{\mathrm{loc}}(A\cup B) \longrightarrow H_k^{\mathrm{loc}}(A) \oplus H_k^{\mathrm{loc}}(B) \longrightarrow H_k^{\mathrm{loc}}(A\cap B) \longrightarrow \ldots$$

where $H_*^{\mathrm{loc}}(Y) := H_*(M, M \smallsetminus Y; R)$. For $k > n$ we obtain $H_k^{\mathrm{loc}}(A \cup B) = 0$. For $k = n$ we have an exact sequence

$$0 \longrightarrow H_n^{\mathrm{loc}}(A\cup B) \longrightarrow H_n^{\mathrm{loc}}(A) \oplus H_n^{\mathrm{loc}}(B) \longrightarrow H_n^{\mathrm{loc}}(A\cap B) \longrightarrow \ldots \qquad (*)$$

and by assumption there are unique classes $\alpha_A \in H_n^{\mathrm{loc}}(A)$, $\alpha_B \in H_n^{\mathrm{loc}}(B)$ and $\alpha_{A\cap B} \in H_n^{\mathrm{loc}}(A\cap B)$ which restrict to $\alpha(x) \in H_n^{\mathrm{loc}}(\{x\})$ for $x \in A$, $x \in B$ and $x \in A \cap B$ respectively. By uniqueness α_A and α_B restrict to $\alpha_{A\cap B}$. Hence, the exact sequence $(*)$ implies that there is a unique class $\alpha_{A\cup B} \in H_n^{\mathrm{loc}}(A \cup B)$ which restricts to $\alpha(x)$ in all points $x \in A \cup B$.

Now, observe that it is enough to prove statements (i) and (ii) for $M = \mathbb{R}^n$: If $A \subset M$ is compact, then $A = A_1 \cup \cdots \cup A_m$ for A_i compact and contained in some coordinate chart of M. So we may assume that $M = \mathbb{R}^n$ and that $A \subset \mathbb{R}^n$ is some compact subset. If additionally A is convex, then for $x \in A$ both $\mathbb{R}^n \smallsetminus \{x\}$ and $\mathbb{R}^n \smallsetminus A$ are both deformation retracts of an $(n-1)$–dimensional sphere centered at x. Therefore, the restriction $H_k^{\mathrm{loc}}(A) \longrightarrow H_k^{\mathrm{loc}}(\{x\})$ is an isomorphism which implies (i) and (ii).

For general compact subsets $A \subset \mathbb{R}^n$, let a class $\alpha_A \in H_k^{\mathrm{loc}}(A)$ be represented by a relative cycle $z \in C_k(M; R)$. Let C be the union of the images of the singular simplices contained in ∂z. Then C is compact and contained in $M \smallsetminus A$. Since A and C are compact and disjoint, they are separated by a positive distance δ. Hence, we can cover A by finitely many closed balls of radius $\delta/2$; denote their union by K. Because $K \cap C = \emptyset$ the chain z defines a class $\alpha_k \in H_k^{\mathrm{loc}}(K)$ which restricts to $\alpha_A \in H_k^{\mathrm{loc}}(A)$. By (ii) for convex subsets we have $H_k^{\mathrm{loc}}(K) = 0$ for $k > n$ and then also $\alpha_A = 0 \in H_k^{\mathrm{loc}}(A)$. If $k = n$, then there is some $\beta_K \in H_n^{\mathrm{loc}}(K)$ restricting to $\alpha(x)$ at all $x \in K$ by statement (i) for compact subsets; restricting

β_K to A yields the existence part in (i) for general A. To prove uniqueness, assume that $\alpha_A \in H_n^{\mathrm{loc}}(A)$ restricts to 0 in all points $x \in A$. By the previous construction, we obtain a class $\alpha_K \in H_n^{\mathrm{loc}}(K)$ restricting to 0 in all points $x \in K$—because $H_n^{\mathrm{loc}}(B) \cong H_n^{\mathrm{loc}}(\{x\})$ for all closed balls B—and we may conclude by the case for convex subsets. □

Remark 6.14. Let $\Gamma(M_R)$ be the R–module of global sections of the covering $M_R \longrightarrow M$. There is a map $H_n(M; R) \longrightarrow \Gamma(M_R)$ which associates to each class α_n the section mapping x to $\rho_x(\alpha_n)$.

If M is connected and R–orientable, then the map $\Gamma(M_R) \longrightarrow H_n(M, M \smallsetminus \{x\}; R) \cong R$ evaluating a global section at x is an isomorphism.

Theorem 6.15. *Let M be a closed topological manifold of dimension n. Then*
(i) *the map $H_n(M; R) \longrightarrow \Gamma(M_R)$ is an isomorphism.*
(ii) *$H_k(M; R) = 0$ for $k > n$.*

Corollary 6.16. *Let M be a closed topological manifold of dimension n. If M is R–orientable, then there exists an R–orientation class $\mu_M \in H_n(M; R)$. If moreover M is connected, then the restriction map $\rho_x \colon H_n(M; R) \longrightarrow H_n(M, M \smallsetminus \{x\}; R)$ is an isomorphism for all $x \in M$.*

Corollary 6.17. *Let M be a closed, connected topological manifold of dimension n. Then there is an isomorphism $H_n(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$ and*

$$H_n(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If M is not orientable, then $\Gamma(M_{\mathbb{Z}}) = 0$, for suppose $0 \neq \alpha \in \Gamma(M_{\mathbb{Z}})$ were some nontrivial section. Then $\alpha(x) \neq 0$ for all $x \in M$ because M is connected. Hence, for all $x \in M$ there is a unique $\mu_x \in H_n(M, M \smallsetminus \{x\}; \mathbb{Z})$ such that $\alpha(x) = m\mu_x$ for some $m \geqslant 1$. This would provide M with an orientation. □

Example 6.18. The manifolds S^n, Σ_g, $\mathbb{CP}^n$ are orientable. The manifolds $\mathbb{RP}^2$ and K^2 are not. More generally, $\mathbb{RP}^n$ is orientable precisely for odd n.

7 Poincaré Duality

If M is a closed R–orientable manifold, then there is an isomorphism $H^k(M;R) \cong H_{n-k}(M;R)$. This section will be devoted to a discussion of this fact.

Definition 7.1. The *cap product* is a bilinear pairing

$$\cap\colon C_k(X;R) \otimes C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$$

satisfying

$$\sigma \cap \varphi = \varphi(\sigma \circ [e_0, \dots, e_\ell]) \cdot \sigma \circ [e_\ell, \dots, e_k]$$

for simplices $\sigma\colon \Delta^k \longrightarrow X$ and cochains $\varphi \in C^\ell(X;R)$.

Lemma 7.2. *The cap product satisfies*

$$\partial(\sigma \cap \varphi) = (-1)^\ell(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi).$$

Proof. We compute

$$\begin{aligned}\partial\sigma \cap \varphi = &\sum_{i=0}^{\ell}(-1)^i\varphi(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{\ell+1}]) \cdot \sigma \circ [e_{\ell+1}, \dots, e_k] + \\ &+ \sum_{i=\ell+1}^{k}(-1)^i\varphi(\sigma \circ [e_0, \dots, e_\ell]) \cdot \sigma \circ [e_\ell, \dots, \widehat{e_i}, \dots, e_k]\end{aligned}$$

and

$$\sigma \cap \delta\varphi = \sum_{i=0}^{\ell+1}(-1)^i\varphi(\sigma \circ [e_0, \dots, \widehat{e_i}, \dots, e_{\ell+1}]) \cdot \sigma \circ [e_{\ell+1}, \dots, e_k].$$

On the left hand side we have

$$\partial(\sigma \cap \varphi) = \sum_{i=\ell}^{k}(-1)^{i-\ell}\sigma \circ [e_0, \dots, e_\ell] \cdot \sigma \circ [e_\ell, \dots, \widehat{e_i}, \dots, e_k].$$

Summing all terms yields the claimed equality. □

Corollary 7.3. *The cap product descends to a pairing*

$$\cap\colon H_k(X;R) \otimes H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R).$$

Remark 7.4.

(i) There are relative versions

$$\cap\colon H_k(X, A;R) \otimes H^\ell(X;R) \longrightarrow H_{k-\ell}(X, A;R)$$

and

$$\cap\colon H_k(X, A;R) \otimes H^\ell(X, A;R) \longrightarrow H_{k-\ell}(X;R)$$

of the cap product.

(ii) For $f\colon X \longrightarrow Y$ one has the naturality

$$f_*\alpha \cap \varphi = f_*(\alpha \cap f^*\varphi)$$

for $\alpha \in H_k(X;R)$ and $\varphi \in H^\ell(X;R)$.

THEOREM 7.5 (Poincaré duality). *Let M be a closed, R–orientable manifold of dimension n. Fix an orientation class $\mu_M \in H_n(M;R)$. Then the map*

$$D\colon H^k(M;R) \xrightarrow{\mu_M \cap \_} H_{n-k}(M;R)$$

is an isomorphism of R–modules.

For the proof of this result we need an additional tool. Let G be an abelian group and denote by $C^k_c(X;R) \subset C^k(X;R)$ the subgroup of cochains φ which admit a compact subset $K \subset X$ such that φ vanishes on chains in $X \setminus K$, i. e. such that $\varphi \in C^k(X, X \setminus K; R)$. Clearly, the boundary operator δ descends to $C^\bullet_c(X;R)$.

DEFINITION 7.6. The cohomology $H^k_c(X;R) = H^k(C^\bullet_c(X;G))$ is called the k^{th} *singular cohomology with compact support.*

REMARK 7.7. Singular cohomology with compact support defines a contravariant functor with respect to *proper* maps $f\colon X \longrightarrow Y$, i. e. continuous maps f such that $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$.

There is an alternative characterisation of compactly supported cohomology.

DEFINITION 7.8. A *directed set* is a partially ordered set I such that for any $i, j \in I$ there is some $k \in I$ such that $i, j \leqslant k$.

DEFINITION 7.9. A *directed system* of abelian groups is a functor $I \longrightarrow$ **Ab** for some directed set I. More explicitly, it is a family $(A_i)_{i\in I}$ of abelian groups together with homomorphisms $\varphi_{ij}\colon A_i \longrightarrow A_j$ for $i \leqslant j$ such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for $i \leqslant j \leqslant k$ and $\varphi_{ii} = \mathrm{id}$.

DEFINITION 7.10. Given a directed system $(A_i)_{i\in I}$, its *direct limit* $\operatorname{colim}_{i\in I} A_i$ is given by the quotient

$$\operatorname*{colim}_{i\in I} A_i = \coprod_{i\in I} A_i \Big/ \sim$$

by the equivalence relation such that $g_i \sim g_j$ if and only if there exists some $k \in I$ with $k \geqslant i, j$ and $\varphi_{ik}(g_i) = \varphi_{jk}(g_j)$. Any two equivalence classes $[g_i]$ and $[g_j]$ have representatives $\varphi_{ik}(g_i)$ and $\varphi_{jk}(g_j)$ respectively in a common group A_k. We set $[g_i] + [g_j] = [\varphi_{ik}(g_i) + \varphi_{jk}(g_j)]$.

There is a natural system of homomorphisms $A_i \longrightarrow \operatorname{colim}_i A_i$ and the direct limit satisfies the following universal property: For any other system of homomorphisms $\psi_i\colon A_i \longrightarrow H$ such that every diagram

$$\begin{array}{ccc} A_i & \longrightarrow & A_j \\ & {\scriptstyle\psi_i}\searrow \quad \swarrow{\scriptstyle\psi_j} & \\ & H & \end{array}$$

commutes, there exists a unique homomorphism $\psi\colon \operatorname{colim}_i A_i \longrightarrow H$ such that every diagram

$$\begin{array}{ccc} A_i & \longrightarrow & \operatorname{colim}_i A_i \\ & \psi_i \searrow \quad \swarrow \psi & \\ & H & \end{array}$$

commutes.

For a topological space X consider the directed set of all compact subsets $K \subset X$. For compact subsets $K \subset L$ there is a natural homomorphism $H^k(X, X \setminus K; G) \longrightarrow H^k(X, X \setminus L; G)$ induced by the inclusion. The inclusions $(X, \emptyset) \subset (X, K)$ for all compact $K \subset X$ induce a compatible system of homomorphisms $H^k(X, X \setminus K; G) \longrightarrow H^k_c(X; G)$. The universal property now supplies us with a natural homomorphism

$$\operatorname*{colim}_{K \subset X \text{ compact}} H^k(X, X \setminus K; G) \longrightarrow H^k_c(X; G).$$

Lemma 7.11. *This map is an isomorphism.*

Similarly, if $X = \bigcup_{i \in I} X_i$ is the union of a directed system $(X_i)_{i \in I}$ of subspaces, then there is a natural homomorphism $\operatorname{colim}_i H_k(X_i; R) \longrightarrow H_k(X; G)$ induced by the inclusions $X_i \longrightarrow X$.

Lemma 7.12. *If every compact subset $K \subset X$ is contained in some X_i, then the natural homomorphism $\operatorname{colim}_i H_k(X_i; R) \longrightarrow H_k(X; R)$ is an isomorphism for all k.*

If M is R–oriented—with a global R–orientation μ—but possibly not compact, we can still define a duality map

$$D_M\colon H^k_c(M; R) \longrightarrow H_{n-k}(M; R)$$

as follows. For a compact subset $K \subset M$ there exists a unique class $\mu_K \in H_n(M, M \setminus K; R)$ restricting to $\mu(x)$ in all points $x \in M$ by proposition 6.13. For $K \subset M$ define a natural homomorphism $H^k(M, M \setminus K; R) \longrightarrow H_{n-k}(M; R)$ by mapping α to $\mu_K \cap \alpha$. In this way we obtain a compatible system of homomorphisms and consequently a unique homomorphism $D_M\colon H^k_c(M; R) \longrightarrow H_{n-k}(M; R)$. For simplicity of notation we will drop the coefficient ring from the notation.

Theorem 7.13. *The duality map $D_M\colon H^k_c(M) \longrightarrow H_{n-k}(M)$ is an isomorphism for all $k \in \mathbb{Z}$.*

Lemma 7.14. *Let $(X, Y) = (A \cup B, C \cup D)$ be a pair of spaces such that $X = A^\circ \cup B^\circ$, $Y = C^\circ \cup D^\circ \subset Y$ and $C \subset A$, $D \subset B$. Then there is a long exact sequence*

$$\ldots \longrightarrow H^n(X, Y) \longrightarrow H^n(A, C) \oplus H^n(B, D) \longrightarrow H^n(A \cap B, C \cap D) \xrightarrow{\delta} H^{n-1}(X, Y) \longrightarrow \ldots$$

which arising as the long exact sequence associated to the short exact sequence

$$0 \longrightarrow C^n(A + B, C + D) \longrightarrow C^n(A, C) \oplus C^n(B, D) \longrightarrow C^n(A \cap B, C \cap D) \longrightarrow 0$$

Lemma 7.15. *If $M = U \cup V$ with $U, V \subset M$ open, then there is a diagram*

$$\begin{array}{ccccccccc} \longrightarrow & H^k_c(U \cap V) & \longrightarrow & H^k_c(U) \oplus H^k_c(V) & \longrightarrow & H^k_c(M) & \longrightarrow & H^{k+1}_c(U \cap V) & \longrightarrow \\ & \big\downarrow D_{U \cap V} & & \big\downarrow D_U - D_V & & \big\downarrow D_M & & \big\downarrow D_{U \cap V} & \\ \longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U \cap V) & \longrightarrow \end{array}$$

with exact rows which commutes up to sign.

Proof. Let $K \subset U$ and $L \subset V$ be compact subsets and write $M \smallsetminus K = A$ and $M \smallsetminus L = B$. Then there is an exact sequence

$$\ldots \longrightarrow H^k(M, A \cup B) \longrightarrow H^k(M, A) \oplus H^k(M, B) \longrightarrow H^k(M, A \cap B) \longrightarrow H^{k+1}(M, A \cup B) \longrightarrow \ldots$$

and the composition of the excision isomorphism and the duality maps yields the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H^k(M, A \cup B) & \longrightarrow & H^k(M, A) \oplus H^k(M, B) & \longrightarrow & H^k(M, A \cap B) & \xrightarrow{\delta} & H^{k+1}(M, A \cup B) & \longrightarrow \\
& \downarrow D_{U \cap V} & & \downarrow D_U - D_V & & \downarrow D_M & & \downarrow D_{U \cap V} & \\
\longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \xrightarrow[\partial]{} & H_{n-k-1}(U \cap V) & \longrightarrow
\end{array}$$

with exact rows. Taking the direct limit, we obtain the result, once we have established that the diagram is commutative up to sign. The only hard part in checking this is the square containing the boundary homomorphisms. So take $[\varphi] \in H^k(M, M \smallsetminus (K \cup L))$ and some $\varphi_A + \varphi_B \in C^k(M, A) \oplus C^k(M, B)$ such that $\varphi_A - \varphi_B = \varphi$. Then $\delta\varphi_A = \delta\varphi_B$ and $\delta[\varphi] = [\delta\varphi_A] \in H^{k+1}(M, M \smallsetminus (K \cap L))$. It remains to compute $\mu_{K \cap L} \cap [\delta\varphi_A]$. Via barycentric subdivision we have $\mu_{K \cap L} = [\alpha]$ with $\alpha = \alpha_{U \smallsetminus L} + \alpha_{U \cap V} + \alpha_{V \smallsetminus K}$ where $\alpha_\bullet \in C_n(\bullet)$. Additionally, $\alpha_{U \cap V}$ represents $\mu_{K \cap L}$ and $\alpha_{U \smallsetminus L} + \alpha_{U \cap V}$ represents μ_K, because for example after restriction of $\mu_{K \cup L}$ the chains $\alpha_{U \smallsetminus L}$ and $\alpha_{V \smallsetminus K}$ become trivial and this restriction is $\mu_{K \cap L}$ by definition.

Therefore, $\mu_{K \cap L} \cap [\delta\varphi_A]$ is represented by $\alpha_{U \cap V} \cap \delta\varphi_A$ which is homologous to $\partial\alpha_{U \cap V} \cap \varphi_A$. Going the other way, the class $\mu_{K \cup L} \cap [\varphi]$ is represented by $\alpha \cap \varphi$ and

$$\partial(\mu_{K \cup L} \cap \varphi) = [\partial(\alpha_{U \smallsetminus L} \cap \varphi)] = \pm(\partial\alpha_{U \smallsetminus L} \cap \varphi - \alpha_{U \smallsetminus L} \cap \delta\varphi) = \pm\partial\alpha_{U \smallsetminus L} \cap \varphi_A$$

because $\partial\alpha_{U \smallsetminus L} \cap \varphi_B = 0$ since φ_B vanishes on chains in B. Now, $\pm\partial\alpha_{U \smallsetminus L} \cap \varphi_A = \mp\partial\alpha_{U \cap V} \cap \varphi_A$ since $\partial(\alpha_{U \smallsetminus L} + \alpha_{U \cap V}) \cap \varphi_A = 0$, for $\alpha_{U \smallsetminus L} + \alpha_{U \cap V}$ represents μ_K and therefore is a relative cycle in $(M, M \smallsetminus K)$. □

Proof of theorem 7.13. Firstly, if $M = U \cup V$ with $U, V \subset M$ open and D_U, D_V and $D_{U \cap V}$ are isomorphisms, then D_U is an isomorphism as well by the previous lemma.

Secondly, if $M = \bigcup_{i \geqslant 1} U_i$ is a union of an increasing chain $U_1 \subset U_2 \subset \ldots$ of open subsets and D_{U_i} is an isomorphism for all i, then D_M is an isomorphism, too: There are natural isomorphisms $H^k_c(U_i) \cong \operatorname{colim}_K H^k(U_i, U_i \smallsetminus K)$ and an isomorphism $\operatorname{colim}_i H^k_c(U_i) \cong H^k_c(M)$.

The theorem can now be proved by first showing the case $M = \mathbb{R}^n$, then for M an open subset of $\mathbb{R}^n$ and then if M is a finite, then countable, then arbitrary union of open subsets, each isomorphic to $\mathbb{R}^n$. We have $H^k_c(\mathbb{R}^n) = H^k(D^n, \partial D^n) = H^k(S^n)$ and in this case the result is easy to see.

Let M be a closed topological manifold of dimension n with a fixed orientation class $\mu_M \in H_n(M; R)$.

THEOREM 7.16. *Every closed topological manifold is homotopy equivalent to a finite CW–complex.*

COROLLARY 7.17. *The homology* $H_*(M)$ *of a closed topological manifold* M *is finitely generated.*

Hence, we may define the Betti numbers $b_i(M) = \operatorname{rk} H_i(M)$ and the Euler characteristic $\chi(M) = \sum(-1)^i b_i(M)$.

THEOREM 7.18. *If* M *is a closed manifold of odd dimension* n*, then* $\chi(M) = 0$.

Proof. First, let M be oriented. By the universal coefficients theorem we have a non–canonical isomorphism $H^k(M;\mathbb{Z}) \cong H_k(M)/H_k(M)_{\text{tor}} \oplus H_{k-1}(M)_{\text{tor}}$ and Poincaré duality yields $H_{n-k}(M) \cong H^k(M;\mathbb{Z})$. Hence, $b_{n-k}(M) = b_k(M)$. This implies the claim for oriented M.

If M is not oriented, we still have $H_{n-k}(M;\mathbb{Z}/2) \cong H^k(M;\mathbb{Z}/2) \cong H_k(M;\mathbb{Z}/2)$ and we may conclude as above, if we prove

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2).$$

Indeed, universal coefficients implies that $H_k(M;\mathbb{Z}/2) \cong H_k(M) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}^1_{\mathbb{Z}}(H_{k-1}(M), \mathbb{Z}/2)$ and the classification of finitely generated abelian groups allows us to conclude. $\square$

For closed manifolds M of dimension n with a chosen orientation class μ_M we may consider the $\cup$–product pairing

$$H^k(M;R) \otimes H^{n-k}(M;R) \longrightarrow R$$

given by the $\varphi \cdot \psi = (\varphi \cup \psi)(\mu_M)$.

Theorem 7.19.
(i) *If* R *is a field, then this pairing is non–degenerate.*
(ii) *If* $R = \mathbb{Z}$*, then the induced pairing*

$$H^k(M;\mathbb{Z})/H^k(M;\mathbb{Z})_{\text{tor}} \otimes H^{n-k}(M;\mathbb{Z})/H^{n-k}(M;\mathbb{Z})_{\text{tor}} \longrightarrow \mathbb{Z}$$

is non–degenerate.

Lemma 7.20. *We have* $(\varphi \cup \psi)(\alpha) = \psi(\alpha \cap \varphi) \in R$.

Proof. For $\psi \in C^k(M;R)$, $\varphi \in C^\ell(M;R)$ and $\sigma\colon \Delta^{k+\ell} \longrightarrow X$ we have

$$\begin{aligned}\psi(\sigma \cap \varphi) &= \psi(\varphi(\sigma \circ [e_0, \dots, e_\ell]) \cdot \sigma \circ [e_\ell, \dots, e_k]) \\ &= \varphi(\sigma \circ [e_0, \dots, e_\ell]) \cdot \psi(\sigma \circ [e_\ell, \dots, e_k]) = (\varphi \cup \psi)(\sigma).\end{aligned}$$

$\square$

Proof of theorem 7.19. By the previous lemma we have $(\varphi \cup \psi)(\mu_M) = \psi(\mu_M \cap \varphi) = \psi(D(\varphi))$. Consider the composition

$$H^{n-k}(M;R) \xrightarrow{h} \operatorname{Hom}_R(H_{n-k}(M), R) \xrightarrow{D^*} \operatorname{Hom}_R(H^k(M;R), R).$$

Here, D^* is an isomorphism by Poincaré duality and if R is a field, then h is an isomorphism. For $R = \mathbb{Z}$ the map h is an isomorphism up to torsion. $\square$

As an application of theorem 7.19 we can compute $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ with α in degree 2 and $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\alpha^{n+1}$ with α in degree 1. For example, assume inductively that α^i generates $H^{2i}(\mathbb{CP}^{n-1};\mathbb{Z})$ for $i \leqslant n-1$. On cohomology, the inclusion $\mathbb{CP}^{n-1} \longrightarrow \mathbb{CP}^n$ induces isomorphisms $H^k(\mathbb{CP}^{n-1},\mathbb{Z}) \cong H^k(\mathbb{CP}^n;\mathbb{Z})$ for $k \leqslant 2n-2$. Now, consider the $\cup$–product pairing

$$H^{2n-2}(\mathbb{CP}^n;\mathbb{Z}) \otimes H^2(\mathbb{CP}^n;\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

It is non-degenerate by theorem 7.19. Therefore $\alpha^{n-1} \cdot \alpha = \pm 1$ which implies that the class α^n generates $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$. The other calculation is similar.

If M has even dimension $2m$, then the $\cup$–product pairing

$$q\colon H^m(M;\mathbb{Z})/H^m(M;\mathbb{Z})_{\text{tor}} \otimes H^m(M;\mathbb{Z})/H^m(M;\mathbb{Z})_{\text{tor}} \longrightarrow \mathbb{Z}$$

is called the *intersection form*. If m is odd, then $\varphi \cup \psi = (-1)^{m^2} \psi \cup \varphi = -\psi \cup \varphi$, i. e. q is skew–symmetric. If m is even, then q is symmetric. In the latter case, we get another invariant $\sigma(M) = \operatorname{sgn}(q)$—the *signature* of M.