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Mathematical Gauge Theory 1

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1 Lie groups and Lie algebras

Definition. A *smooth/differentiable manifold* M is a topological space with the property that

(1) M is Hausdorff

(2) M is second countable, i.e. there is a countable basis of its topology

(3) There exists a covering of M by open U_i, $i \in I$, and homeomorphisms $\varphi_i \colon U_i \to \mathbb{R}^n$ such that when $U_i \cap U_j \neq \varnothing$, then $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a diffeomorphism.

If $f \colon M \to N$ is a smooth map between smooth manifolds, then there is a derivative $Df \colon TM \to TN$. Df is linear on the fibers of TM, i.e. $D_p f \colon T_pM \to T_{f(p)}N$ is linear, and the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{Df} & TN \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & N \end{array}$$

$\mathfrak{X}(M) \coloneqq \{X \colon M \to TM \mid X \text{ is smooth} \wedge \pi \circ X = \mathrm{id}_M\}$ is the set of smooth sections of TM, i.e. vector fields on M. For $f \in C^\infty(M)$ we have $L_X f \in C^\infty(M)$, the Lie derivative of f in the direction of X. $[X, Y]$ is the unique vector field with the property $L_{[X,Y]} f = L_X L_Y f - L_Y L_X f$.

Definition. A (real) *Lie algebra* is an $\mathbb{R}$-vector space $\mathfrak{g}$ together with a bilinear map $[-,-] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

(1) $[v, w] = -[w, v]$

(2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

Example.

(1) $\mathfrak{X}(M)$ with $[-,-]$ defined by the Lie bracket is an ∞-dimensional Lie algebra.

(2) Let V be any $\mathbb{R}$-vector space. Then $[-,-] = 0$ defines a Lie algebra. These are abelian Lie algebras.

(3) $\mathrm{Mat}(n \times n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$, $[A, B] = AB - BA$.

Definition. A *Lie group* is a group which is also a smooth manifold and has the property that $G \times G \to G, (a, b) \mapsto ab^{-1}$ is smooth.

Example.

(1) Any finite dimensional $\mathbb{R}$-vector space V with the group structure given by $+$.

(2) $\mathrm{GL}(n, \mathbb{R})$

(3) subgroups of $\mathrm{GL}(n,\mathbb{R})$: $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$.

A Lie group G acts on itself by left multiplication. For any $g \in G$, $\ell_g \colon G \to G, a \mapsto ga$ is a smooth map. ℓ_g is a diffeomorphism with inverse $(\ell_g)^{-1} = \ell_{g^{-1}}$.

Definition. A vector field $X \in \mathfrak{X}(G)$ is called *left–invariant* if $X_{ga} = X_{\ell_g(a)} = (D_a\ell_g)X_a$.

Proposition. *The subset of left–invariant vector fields in $\mathfrak{X}(G)$ is a linear subspace closed under the Lie bracket $[-,-]$. Thus it is a Lie algebra.*

Proof. X left invariant means $(D\ell_g)X = X \circ \ell_g$ for all $g \in G$. Assume that X, Y are both left invariant. Then $[X,Y] = [(D\ell_g)X, (D\ell_g)Y] = (D\ell_g)[X,Y]$ for all $g \in G$. So $[X,Y]$ is also left–invariant. ☐

Definition. $\mathfrak{X}(G)$ with the Lie bracket $[-,-]$ is the *Lie algebra* $\mathfrak{g} = \mathrm{L}(G)$ *of the Lie group* G.

Convention. From now on, G, H are always Lie groups and $\mathfrak{g} = \mathrm{L}(G)$, $\mathfrak{h} = \mathrm{L}(H)$ its Lie algebras.

Definition. Let G be a Lie group with neutral element $e \in G$. Define $\mathrm{ev}\colon \mathfrak{g} \to T_eG, X \mapsto X_e$.

Proposition. ev *is an isomorphism of* $\mathbb{R}$*–vector spaces.*

Proof.
(1) ev is clearly linear.

(2) ev is injective: Suppose $X, Y \in \mathfrak{g}$ and $\mathrm{ev}(X) = \mathrm{ev}(Y)$. This means $X_e = Y_e \Rightarrow (D_e\ell_g)X_e = (D_e\ell_g)Y_e \Rightarrow X_g = Y_g \Rightarrow X = Y$.

(3) ev is surjective: Take $v \in T_eG$. Define $X \in \mathfrak{X}(G)$ by $X_g \coloneqq (D_e\ell_g)(v)$. This is a smooth vector field and $X_{ga} = (D_e\ell_{ga})(v) = (D_e(\ell_g \circ \ell_a))(v) = (D_a\ell_g \circ D_e\ell_a)(v) = (D_a\ell_g)(X_a)$. ☐

Corollary. *The dimension of G is constant and equals the dimension of $\mathfrak{g}$ as $\mathbb{R}$–vector space.*

Proof. Denoting by G_g the connected component of G containing $g \in G$, $\ell_g \colon G_e \to G_g$ is a diffeomorphism for every $g \in G$. Thus all connected components have the same dimension and $\dim G_0 = \dim T_eG = \dim \mathfrak{g}$. ☐

Corollary. *The tangent bundle of G is globally trivial, i.e. G is parallelizable.*

Proof. Consider $G \times \mathfrak{g} \to TG, (g,X) \mapsto X_g$. This sends the fiber of $G \times \mathfrak{g}$ over $g \in G$ to T_gG linearly. At the point e we have $(e,X) \mapsto X_e$, which is an isomorphism by the result about the evaluation map. For arbitrary $g \in G$ we have:

$$\begin{array}{ccc} \{g\} \times \mathfrak{g} & \longrightarrow & T_gG \\ \simeq \Big\uparrow \ell_g \times \mathrm{id}_{\mathfrak{g}} & & \simeq \Big\uparrow D_e\ell_g \\ \{e\} \times \mathfrak{g} & \longrightarrow & T_eG \end{array}$$

This implies that the top horizontal map is an isomorphism, as claimed. ☐

Definition. A *one–parameter subgroup* of G is a smooth map $s\colon \mathbb{R} \to G$ with $s(0) = e$ and $s(t_1 + t_2) = s(t_1)s(t_2)$.

Example. Consider $G = U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Then $s\colon \mathbb{R} \to U(1), t \mapsto e^{2\pi i t}$ is a (not injective) one–parameter subgroup.

Proposition.

(1) Every left–invariant vector field on G is complete, i.e. it generates a global flow on G.

(2) For every $X \in \mathfrak{g}$ there is a unique one–parameter subgroup $s_X\colon \mathbb{R} \to G$ such that $\dot{s}_X(0) \coloneqq D_0 s_X(\partial_t) = X_e$. The flow of X is given by $\varphi\colon \mathbb{R} \times G \to G, (t, g) \mapsto g s_X(t) = \ell_g(s_X(t))$.

Notation. We write $\varphi_t(g) \coloneqq \varphi(t, g)$.

Proof. Given a left–invariant vector field, there is a local flow at $e \in G$:

$$\varphi\colon (-\varepsilon, \varepsilon) \times U \longrightarrow G$$

where U is an open neighbourhood of e in G, $\varepsilon > 0$. For any $g \in G$ consider

$$\psi\colon (-\varepsilon, \varepsilon) \times \ell_g(U) \longrightarrow G \qquad \psi \coloneqq \ell_g \circ \varphi \circ (\mathrm{id}_{\mathbb{R}} \times \ell_g^{-1}) \quad \Rightarrow \quad \psi_t = \ell_g \circ \varphi_t \circ \ell_g^{-1}$$

Claim: ψ is a local flow around g for X. Proof: $\ell_g(U)$ is an open neighbourhood of g in G. First we check that ψ defines a local flow:

$$\psi_0(h) = g\varphi_0(g^{-1}h) = gg^{-1}h = h$$

$$\psi_{t_1+t_2} = \ell_g \circ \varphi_{t_1+t_2} \circ \ell_g^{-1} = \ell_g \circ \varphi_{t_1} \circ \varphi_{t_2} \circ \ell_g^{-1} = \ell_g \circ \varphi_{t_1} \circ \ell_g^{-1} \circ \ell_g \circ \varphi_{t_2} \circ \ell_g^{-1} = \psi_{t_1} \circ \psi_{t_2}$$

So ψ is a flow generated by some vector field, which we can calculate by differentiating the flow lines of ψ. To do this, consider the flow lines defined by

$$s_h(t) \coloneqq \psi_t(h) = g\varphi_t(g^{-1}h) = (\ell_g \circ s_{g^{-1}h})(t)$$

Then

$$\begin{aligned}\dot{s}_g(0) &= D_0 s_g(\partial_t) = D_0(\ell_g \circ s_e)(\partial_t) = (D_{s_e(0)}\ell_g \circ D_0 s_e)(\partial_t) = (D_{\varphi_0(e)}\ell_g \circ D_0 s_e)(\partial_t) = \\ &= D_e\ell_g(D_0 s_e(\partial_t)) = D_e\ell_g(\dot{s}_e(0)) = D_e\ell_g(X_e) = X_g\end{aligned}$$

This proves the claim.

These local flows defined at different points in G are all defined for the same time interval $(-\varepsilon, \varepsilon)$, and so define a flow $\overline{\varphi}\colon (-\varepsilon, \varepsilon) \times G \to G$ for X. $\overline{\varphi}$ can be extended to all $t \in \mathbb{R}$, so X is complete.

To prove part (2) fix $X \in \mathfrak{g}$. By (1) we have a global flow $\varphi\colon \mathbb{R} \times G \to G$ for X. Define $s_X(t) \coloneqq \varphi_t(e)$. Since

$$\varphi_t(g) = (\ell_g \circ \varphi_t \circ \ell_g^{-1})(g) = \ell_g(\varphi_t(e)) = g\varphi_t(e)$$

we have $s_X(0) = \varphi_0(e) = e$ and

$$s_X(t_1 + t_2) = \varphi_{t_1+t_2}(e) = \varphi_{t_2+t_1}(e) = \varphi_{t_2}(\varphi_{t_1}(e)) = \varphi_{t_1}(e)\varphi_{t_2}(e) = s_X(t_1)s_X(t_2)$$

Also $\dot{s}_X(0) = X_e$, since s_X is the flow line at e, and the formula $\varphi_t(g) = gs_X(t)$ follows from the claim above. One can easily check that φ defined by this formula is a global flow for any one–parameter subgroup s_X and thus s_X is unique by the uniqueness of global flows. □

Definition. The map $\exp\colon \mathfrak{g} \to G, X \mapsto s_X(1)$ is the *exponential map* of G.

Example. Let $G = \mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}(n \times n, \mathbb{R})$. Then exp is given by the usual exponential function.

Lemma. $s_X(t) = \exp(tX)$.

Proof. Clear.

Definition. A homomorphism of Lie groups is a smooth map $f\colon G \to H$ which is also a group homomorphism.

Proposition. *Any homomorphism $f\colon G \to H$ as above induces an $\mathbb{R}$–linear homomorphism $f_*\colon \mathfrak{g} \to \mathfrak{h}$ such that $f_*[X, Y] = [f_*X, f_*Y]$.*

Proof of proposition. Since f is a smooth group homomorphism, it has a derivative at e and $f(e) = e$.

$$\begin{array}{ccc} T_eG & \xrightarrow{D_ef} & T_eH \\ \simeq \uparrow \mathrm{ev} & & \simeq \uparrow \mathrm{ev} \\ \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \end{array}$$

Under the homomorphism provided by ev, D_ef corresponds to a unique linear f_*.

Lemma. *For any $X \in \mathfrak{g}$ and $g \in G$ we have $(D_gf)(X_g) = (f_*X)_{f(g)}$.*

Proof. Direct calculation using the left–invariance of X and $f(e) = e$.

$$\begin{aligned}(D_gf)(X_g) &= D_gf \circ D_e\ell_g(X_e) = D_e(f \circ \ell_g)(X_e) = D_e(\ell_{f(g)} \circ f)(X_e) = \\ &= D_e\ell_{f(g)} \circ D_ef(X_e) = D_e\ell_{f(g)}((f_*X)_e) = (f_*X)_{f(g)}\end{aligned}$$ □

Using this lemma, we can show that $f_*X(h) \circ f = X(h \circ f)$ for any $X \in \mathfrak{X}(M)$ and $h \in C^\infty(H)$:

$$(f_*X)(h)(f(g)) = (f_*X)_{f(g)}[h]_{f(g)} = (D_gf)(X_g)[h]_{f(g)} = X_g[h \circ f]_g = X(h \circ f)(g)$$

Thus, applying this successively:

$$\begin{aligned}[f_*X, f_*Y](h) \circ f &= f_*X(f_*Y(h)) \circ f - f_*Y(f_*X(h)) \circ f = \\ &= X(f_*Y(h) \circ f) - Y(f_*X(h) \circ f) = \\ &= X(Y(h \circ f)) - Y(X(h \circ f)) = \\ &= [X, Y](h \circ f) = f_*[X, Y](h) \circ f\end{aligned}$$

Since this holds for any $h \in C^\infty(H)$, we have $[f_*X, f_*Y] \circ f = f_*[X,Y] \circ f$, so in particular $[f_*X, f_*Y]$ and $f_*[X,Y]$ coincide on $e \in H$, hence they are equal by left–invariance, which proves the proposition. □

1.1 Digression on integrability and the Frobenius theorem

Theorem. *Let M be a smooth manifold and $X_1, \ldots, X_k \in \mathfrak{X}(M)$ with $[X_i, X_j] = 0$ for all i, j. If $X_1(p), \ldots, X_k(p)$ are linearly independent for some $p \in M$, then there is a chart (U, φ) for M with $p \in U$ and $D\varphi(X_i|U) = \partial_i$ for all $i = 1, \ldots, k$.*

Proof. The problem is local, so we may assume M is an open neighborhood of 0 in $\mathbb{R}^n$ and $p = 0$. We may choose U open around 0 with $X_1, \ldots, X_k$ linearly independent throughout U. After a linear change of coordinates we may assume that

$$X_1(0), \ldots, X_k(0), \partial_k(0), \ldots \partial_n(0)$$

form a basis of $\mathbb{R}^n$. We may assume that the local flow φ^i for X_i is defined for all $t \in (-\varepsilon, \varepsilon)$, $i = 1, \ldots, k$.

$$f \colon U \to \mathbb{R}^n \qquad f(x_1, \ldots, x_n) = \varphi^1_{x_1} \circ \cdots \circ \varphi^k_{x_k}(0, \ldots, 0, x_{k+1}, \ldots, x_k)$$

is a smooth map. Moreover $f(0) = 0$ and $D_0 f(\partial_i) = \partial_i$ for $i = k+1, \ldots, n$. For all $x \in U$ we have, since the flows φ^i commute:

$$D_x f(\partial_i) = X_i(f(x)) \qquad i = 1, \ldots, k$$

For $x = 0$ we see that $D_0 f$ is an isomorphism, so f is a local diffeomorphism around 0 by the inverse function theorem. Set $\varphi = f^{-1}$ after possibly shrinking U:

$$D\varphi(X_i|f(U)) = \partial_i$$

□

Let M be a smooth manifold of dimension n.

Definition. A *rank k distribution* on M is a rank k subbundle $E \subset TM$.

What this means is that around every point $p \in M$ there exists an open set U and $X_1, \ldots, X_k \in \mathfrak{X}(M)$ such that

$$E_x = \{X_1(x), \ldots, X_k(x)\}$$

Definition. An *integral submanifold* for E is a k–dimensional submanifold $N \subset M$ with $TN = E|N$.

Definition. E is called *integrable* if for all $p \in M$ there exists an integrable submanifold N with $p \in N$.

Definition. E is *involutive* if $[X, Y] \in \Gamma(E)$ whenever $X, Y \in \Gamma(E)$.

Theorem (Frobenius theorem)**.** *For a distribution E of rank k on M, the following are equivalent:*

(1) E is integrable.

(2) E is involutive.

(3) There is a covering of M by domains of charts (U, φ) with the property that

$$(D\varphi)(E) \ni \partial_i \qquad \forall i = 1, \ldots, k$$

Proof.

3 ⇒ 1 If $E = \operatorname{span}\{\partial_1, \ldots, \partial_k\}$, then the equations

$$\begin{aligned} x_{k+1} &= c_{k+1} \\ &\vdots \\ x_n &= c_n \end{aligned}$$

define k–dimensional submanifolds for E.

1 ⇒ 2 Take $X, Y \in \Gamma(E)$ and $p \in M$. By (1) we have a submanifold $i\colon N \hookrightarrow M$ with $p \in N$ and $E|N = TN$. The restrictions of X, Y to N are vector fields on N. Furthermore, $[X, Y]_p \in E_p$.

2 ⇒ 3 Everything is local, so we work at $0 \in \mathbb{R}^n$.

Step 1: Consider the projection

$$\pi\colon \mathbb{R}^n \to \mathbb{R}^k, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$$

If for some point p, $D_p\pi$ is injective on E_p, then the same is true for all x in an open neighbourhood of p.

Step 2: At every point p there is a chart so that w.r.t. the coordinates given by the chart $D_x\pi$ is an isomorphism from E_x to $\mathbb{R}^n$ for all x in the domain of the chart. To prove this, by step 1 it is enough to ensure $D_p\pi$ is injective. We can choose local coordinates $(x_1, \ldots, x_k)$ in such a way that if

$$X_1(p), \ldots, X_k(p)$$

is a basis for E_p, then

$$X_1(p), \ldots, X_k(p), \partial_{k+1}, \ldots, \partial_n$$

is a basis of $T_p\mathbb{R}^n$.

Step 3: Let E, p, U, π be as above. Let $Z_i \in \Gamma(E)$ be the unique section such that $D_x\pi(Z_i(x)) = \partial_i(x)$ for all $x \in U$, $i = 1, \ldots, k$. So $Z_1, \ldots, Z_k$ span E throughout U.

Step 4:
$$D\pi[Z_i, Z_j] = [D\pi Z_i, D\pi Z_j] = [\partial_i, \partial_j] = 0$$

By involutivity, $[Z_i, Z_j] \in \Gamma(E|U)$. But $D\pi$ is an isomorphism on E so $[Z_i, Z_j] = 0$. By the previous theorem, we can find a chart in which $D\varphi(Z_i) = \partial_i$ for $i = 1, \ldots, k$. This gives (3) in the Frobenius theorem. □

Definition. If E satisfies the conditions in the theorem above and $p \in M$, let L_p be the maximal connected submanifold of E with $p \in L_p$. This is called the *leaf* through p. The collection of all leaves formes a *foliation* of M.

Remark. The leaves of a foliation $\mathcal{F}$ are generally not closed subsets of M and the subspace topology is not the same as the manifold topology of a leaf.

Definition. Let G be a Lie group, $H \subset G$ a subset. H is a *Lie subgroup* of G if H has a Lie group structure such that the inclusion $i\colon H \hookrightarrow G$ is a homomorphism of Lie groups and an injective immersion.

Theorem. *For any Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$.*

Proof. Suppose $H \subset G$ is a Lie subgroup. Then, since i is an immersion,

$$D_e i\colon T_eH \to T_eG \quad \text{and} \quad i_*\colon \mathfrak{h} \to \mathfrak{g},$$

which are essentially the same maps, are injective. So $i_*(\mathfrak{h})$ is a Lie subalgebra, which can be identified with $\mathfrak{h}$.

Conversely, let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Let $E_g := D_e\ell_g(\mathrm{ev}(\mathfrak{h})) \subset T_gG$ for all $g \in G$. This is in fact the evaluation of $\mathfrak{h}$ at g. For every $g \in G$, E_g is a k–dimensional subspace of T_gG with $k = \dim \mathfrak{h}$. The collection of all E_g is a smooth rank k distribution $E \subset TG$.

Step 1: E is involutive, and thus integrable by the Frobenius theorem. To see this, let $X_1, \ldots, X_k$ be a basis for $\mathfrak{h}$. Then all $X, Y \in \Gamma(E)$ are of the form

$$X = \sum_{i=1}^{k} f_i X_i \qquad Y = \sum_{i=1}^{k} h_i X_i \qquad f_i, h_i \in C^\infty(G)$$

Then $[X, Y]$ is a linear combination of the X_i and the $[X_i, X_j]$. Since $\mathfrak{h}$ is a Lie subalgebra, $[X_i, X_j] \in \mathfrak{h}$ and so $[X, Y] \in \Gamma(E)$.

Step 2: Let $\mathcal{F}$ be the foliation of G defined by the integral submanifolds of E, and $H := L_e$. Then $L_g = \ell_g(H)$. Proof: Both sides are connected subsets containing g. L_g is a leaf of $\mathcal{F}$ by definition. Once we prove that $\ell_g(H)$ is a leaf, we have the conclusion by the uniqueness of leaves. For any $a \in G$, $b \in H$ we have

$$T_{ab}\ell_a(H) = D_b\ell_a(T_bH) = D_b\ell_a(E_b) = (D_b\ell_a \circ D_e\ell_b)(\mathrm{ev}(\mathfrak{h})) = D_e\ell_{ab}(\mathrm{ev}(\mathfrak{h})) = E_{ab}$$

so $\ell_a(H)$ is an integral submanifold of E and thus $g \cdot H := \ell_g(H)$ is the leaf of $\mathcal{F}$ through g.

Step 3: Let $a, b \in H$. Then $aH = H \ni b$, since aH is the leaf through a. But $b \in aH$, implies $a^{-1}b \in H$, so H is a subgroup of G.

Step 4: The inclusion $i\colon H \hookrightarrow G$ makes H into a Lie subgroup. Since $D_g i(T_g H) = E_g$, i is an injective immersion, with the manifold structure of H given by its construction as an integral submanifold for E.

Step 5: This H is the only connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. To prove this, suppose $\overline{H}$ is another Lie subgroup with Lie algebra $\mathfrak{h}$. Both $H, \overline{H}$ are injectively immersed in G with the same tangent space $\mathrm{ev}(\mathfrak{h})$ at e. $\overline{H}$ will then be an integral submanifold for E through e. By uniqueness of the leaf through e, we must have $H = \overline{H}$. □

Example. The 2–torus $G = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a connected Lie group with $e = [(0,0)]$. The Lie algebra is $\mathfrak{g} = \mathbb{R}^2$ with $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Every vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra in this case, giving rise to a Lie subgroup. If $\mathfrak{h} = \mathrm{span}(1, \lambda)$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then the corresponding connected Lie subgroup is

$$H = \{\exp(t(1, \lambda)) \mid t \in \mathbb{R}\}$$

This is densely immersed in T^2, in particular it is not a closed subgroup.

1.2 Actions of Lie groups on manifolds

Definition. A *(left) action* of a Lie group G on a smooth manifold M is a smooth map

$$\mu\colon G \times M \to M \qquad \mu(g, p) = g \cdot p = \ell_g(p)$$

such that for any $p \in M$ and $g, h \in G$

$$e \cdot p = p \qquad g \cdot (h \cdot p) = (gh) \cdot p$$

A *right action* is a smooth map $\mu\colon G \times M \to M$ such that for any $p \in M$ and $g, h \in G$

$$\mu(e, p) = p \qquad \mu(g, \mu(h, p)) = \mu(hg, p)$$

For a right action, we write $\mu(g, p) = p \cdot g = r_g(p)$. Then the axioms become

$$p \cdot e = p \qquad (p \cdot h) \cdot g = p \cdot (hg)$$

Remark. If $\mu\colon G \times M \to M$ is a left action, we can define $\overline{\mu}(g, p) = \mu(g^{-1}, p)$. This is a right action.

Definition. Let $\mu\colon G \times M \to M$ be a an action of a Lie group on a smooth manifold.

(1) μ is *effective* if for every $g \in G \setminus \{e\}$, there exists $p \in M$ such that $\mu(g, p) \neq p$.

(2) For $p \in M$, the subset

$$G(p) := \{\mu(g, p) \mid g \in G\}$$

is the *orbit* of p under the action.

(3) The action is *transitive* if $G(p) = M$ for some $p \in M$ (and thus for all $p \in M$).

(4) The *isotropy group* of $p \in M$ is

$$G_p \coloneqq \{g \in G \mid \mu(g,p) = p\}$$

Proposition. *Let $f\colon G \to H$ be a homomorphism of Lie groups and $f_*\colon \mathfrak{g} \to \mathfrak{h}$ the induced Lie algebra homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\;f\;} & H \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\;f_*\;} & \mathfrak{h} \\ {\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathrm{ev}} \\ T_eG & \xrightarrow[D_ef]{} & T_eH. \end{array}$$

Proof. $f(\exp(tX))$ is a C^∞ curve in H passing through e at time $t = 0$. Since f is a homomorphism, this is also a 1–parameter subgroup whose tangent vector at e is $D_ef(\mathrm{ev}(X)) = \mathrm{ev}(f_*(X))$. So $f(\exp(tX))$ is the unique 1–parameter subgroup of H generated by $f_*X = Y$. We know that the 1–parameter subgroup generated by Y is $\exp(tY)$. Therefore

$$f(\exp(tX)) = \exp(tY) = \exp(tf_*X) \quad \xRightarrow{t=1} \quad f(\exp(X)) = \exp(f_*X) \qquad \square$$

The isotropy group G_p is a closed subgroup of G. This means that G_p is actually a Lie subgroup of G (not proved). If we restrict μ from G to G_p, then p is a fixed point for the action of G_p on M.

Under an action $\mu\colon G \times M \to M$, every $g \in G$ gives a diffeomorphism

$$\ell_g\colon M \to M, p \mapsto g \cdot p$$

with inverse $(\ell_g)^{-1} = \ell_{g^{-1}}$.

Lemma. *If p is a fixed point of the action $\mu\colon G \times M \to M$, then G acts linearly on T_pM, so we have a representation $G \to \mathrm{GL}(T_pM)$. This is called the* isotropy representation *of G at p.*

Proof. Since p is a fixed point, we have $\ell_g(p) = p$ for all $g \in G$, so $D_p\ell_g\colon T_pM \to T_pM$ is a linear isomorphism since ℓ_g is a diffeomorphism. We obtain a map

$$G \to \mathrm{GL}(T_pM), g \mapsto D_p\ell_g$$

which is smooth since μ is smooth. It is also a homomorphism because of the chain rule:

$$D_p\ell_{g_1g_2} = D_p(\ell_{g_1} \circ \ell_{g_2}) = D_p\ell_{g_1} \circ D_p\ell_{g_2} \qquad \square$$

Example. The action

$$\mu\colon G \times G \to G, (g, p) \mapsto g \cdot p = \ell_g(p)$$

is effective, transitive, and $G_p = \{e\}$ for all $p \in G$.

Let G act on itself by conjugation:

$$a\colon G \times G \to G, (g, p) \mapsto g \cdot p \cdot g^{-1} =: a_g(p)$$

Note that g and p commute in G if and only if $a_g(p) = p$. The isotropy group G_p of a point $p \in G$ under the conjugation action a is the centraliser of p in G. If G has a non–trivial center, then the conjugation action a is not effective. One has $G_e = G$ since $geg^{-1} = e$ for all $g \in G$. By the Lemma, we obtain the isotropy representation of a at $p = e$:

$$\mathrm{Ad}\colon G \to \mathrm{GL}(T_eG).$$

This is the *adjoint representation* of G (on $\mathfrak{g}$), whereas the map ad defined by

$$\begin{array}{ccc} T_eG & \xrightarrow{D_e\,\mathrm{Ad}} & T_e\,\mathrm{GL}(T_eG) \\ \mathrm{ev}\big\uparrow\simeq & & \simeq\big\uparrow\mathrm{ev} \\ \mathfrak{g} & \xrightarrow[\mathrm{ad}]{} & \mathrm{End}(\mathfrak{g}) \end{array}$$

with the identification $\mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(T_eG)$ is the *adjoint representation* of $\mathfrak{g}$. Since $\mathrm{ad} = \mathrm{Ad}_*$, it is a homomorhism of Lie algebras. By the proposition, we have the commutative diagram:

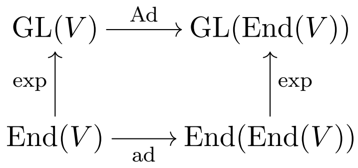
$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(T_eG) \\ \mathrm{exp}\big\uparrow & & \big\uparrow\mathrm{exp} \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathfrak{g}) \end{array}$$

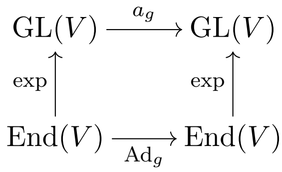
Also a_g is a homomorphism of G to itself, so the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{a_g} & G \\ \mathrm{exp}\big\uparrow & & \big\uparrow\mathrm{exp} \\ \mathfrak{g} & \xrightarrow{(a_g)_*} & \mathfrak{g} \\ \mathrm{ev}\big\downarrow\simeq & & \simeq\big\downarrow\mathrm{ev} \\ T_eG & \xrightarrow[D_ea_g]{} & T_eG \end{array}$$

Notation. Define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_X(Y) = \mathrm{ad}(X)(Y)$ for any $X, Y \in \mathfrak{g}$ and $\mathrm{Ad}_g : T_eG \to T_eG$ by $\mathrm{Ad}_g(X) = \mathrm{Ad}(g)(X)$ for any $g \in G$ and $X \in T_eG$.

Then $(a_g)_* = \mathrm{Ad}_g = \mathrm{Ad}(g)$, since $(a_g)_*$ is defined by $D_e a_g = \mathrm{Ad}(g)$ by the definition of Ad. Take a vector space V and $G = \mathrm{GL}(V)$. Then the above diagrams become:





We claim that $\mathrm{Ad}_g(M) = gMg^{-1}$. This can be seen by

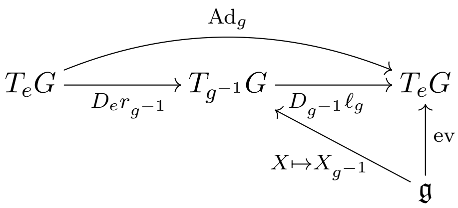
$$\mathrm{Ad}_g(M) = D_e a_g(M) =$$

(something is missing here)

Consider $r_{g^{-1}} \colon G \to G$, the right multiplication by g^{-1}. Recall that $TG \cong G \times \mathfrak{g}$ is a trivialization of TG given by

$$G \times \mathfrak{g} \to TG \quad (p, X) \mapsto (p, X_p)$$

Lemma. $\mathrm{Ad}_g \in \mathrm{GL}(T_e G)$ *is given by the composition of* $D_e r_{g^{-1}}$ *with the identification of* $T_{g^{-1}}G$ *with* $T_e G$ *via the trivialization of the tangent bundle by left–invariant vector fields.*



Proof. Any tangent vector $v \in T_{g^{-1}}G$ can be identified with $X_e \in T_e G$ for the unique $X \in \mathfrak{g}$ such that $X_{g^{-1}} = v$. This identification is via $D_{g^{-1}}\ell_g$:

$$D_{g^{-1}}\ell_g(v) = D_{g^{-1}}\ell_g(X_{g^{-1}}) = D_{g^{-1}}\ell_g \circ D_e \ell_{g^{-1}}(X_e) = X_e$$

Using this, Ad_G we get the claim:

$$D_{g^{-1}}\ell_g \circ D_e r_{g^{-1}} = D_e a_g = \mathrm{Ad}_g \qquad \square$$

Definition. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

$$C(G) := \{g \in G \mid gh = hg \, \forall h \in G\}$$

is the *center* of G and

$$C(\mathfrak{g}) := \{X \in \mathfrak{g} \mid [X, Y] = 0 \, \forall Y \in \mathfrak{g}\}$$

is the *center* of $\mathfrak{g}$.

Lemma. *Let G be a connected Lie group. Then* $\ker \mathrm{Ad} = C(G)$.

Proof. If $g \in C(G)$, then $a_g = \mathrm{id}_g$, so $\mathrm{Ad}_g = D_e a_g = \mathrm{id}_{T_eG}$, so $g \in \ker \mathrm{Ad}$. Conversely, suppose that $g \in \ker \mathrm{Ad}$. Then

$$g \exp(tX) g^{-1} = a_g(\exp(tX)) = \exp(\mathrm{Ad}_g(tX)) = \exp(tX)$$

for all $X \in T_eG$. So g commutes with all $h \in G$ containted in a small enough neighbourhood in G, because exp is a local diffeomorphism at $0 \in T_eG$. Every open neighbourhood of e in G generates the connected component of e in G by taking products. Therefore, if G is connected, then $g \in C(G)$. □

Corollary. *Let G be a connected Lie group. Then $C(G)$ is a closed Lie subgroup whose Lie algebra is the center of $\mathfrak{g}$.*

Proof. The center $C(G)$ is a closed subgroup of G. So it is a Lie subgroup. The Lie algebra of $C(G)$ is $C(\mathfrak{g})$. □

Corollary. *If G is a connected Lie group, then G is abelian iff $\mathfrak{g}$ has trivial Lie brackets.*

1.3 Homogeneous spaces

Let G be a Lie group, $H \subset G$ a closed Lie subgroup. Consider

$$G/H \coloneqq \{aH \mid a \in G\},$$

the set of left cosets of H. We denote by

$$\ell_g \colon G/H \to G/H \quad aH \mapsto (ga)H$$

the action induced by left multiplication. For any two $a, b \in G$, there exists $g \in G$ such that $\ell_g(aH) = bH$.

Theorem. *G/H has a natural structure as a smooth manifold of dimension* $\dim G - \dim H$*, such that*

$$\pi \colon G \to G/H \quad a \mapsto aH$$

is a smooth map that admits local smooth sections. π will actually be a submersion.

Proof. This proof will be added later

Corollary. *The action*

$$\mu \colon G \times G/H \to G/H \quad (g, aH) \mapsto (ga)H$$

defines a transitive smooth action of G on G/H. The isotropy group of $H = eH$ is H.

Proof. μ is a smooth map by the construction of the smooth structure on G/H. μ is a left action of G. The action is transitive, and

$$G_{eH} = \{g \in G \mid \mu(g, eH) = eH\} = H$$ □

2 Principal bundles

Definition. A *principal G–bundle* over a smooth manifold M is a smooth manifold P, a smooth projection $\pi\colon P \to M$ and a right G–action $\cdot\colon P \times G \to P$ satisfying:

(a) There is a covering of M by open sets U_α, together with diffeomorphisms

$$\pi^{-1}(U_\alpha) \to U_\alpha \times G \qquad p \mapsto (\pi(p), \varphi_\alpha(p))$$

(b) For all $g \in G$, $p \in \pi^{-1}(U_\alpha)$, we have $\varphi_\alpha(r_g(p)) = r_g(\varphi_\alpha(p))$, i.e. $\varphi_\alpha(pg) = \varphi_\alpha(p)g$ (G–equivariance)

G is called the *structure group* of P.

Remark.

(1) π is a submersion because $\pi^{-1}(U_\alpha) \cong U_\alpha \times G$ and π_1 is a submersion.

(2) $\pi^{-1}(m) \cong G$ for all $m \in M$.

(3) The action of G on P maps $\pi^{-1}(m)$ to itself for all $m \in M$.

(4) On each fiber $\pi^{-1}(m)$, the G–action is simply transitive, i.e. transitive and has trivial stabilizers. It follows also that G is free on the whole principal bundle P.

Let $X \in \mathfrak{g}$, then $\exp(tX)$ is a one–parameter subgroup of G. By restricting the right action $P \times G \to P$ we obtain a flow on P, which is generated by some vector field $X^* \in \mathfrak{X}(P)$.

Definition. X^* is the *fundamental vector field* generated by X. X^* is tangent to the fiber of π.

Lemma. *For any $g \in G$, $X \in \mathfrak{g}$ and $p \in P$, we have*

$$D_p r_g(X^*_p) = (\mathrm{Ad}_{g^{-1}}(X))^*_{pg}$$

Proof. We have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{a_{g^{-1}}} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\mathrm{Ad}_{g^{-1}}]{} & \mathfrak{g} \end{array}$$

so, defining $Y \coloneqq \mathrm{Ad}_{g^{-1}}(X)$: $\exp(tY) = \exp(t\,\mathrm{Ad}_{g^{-1}}(X)) = \exp(\mathrm{Ad}_{g^{-1}}(tX)) = g^{-1}\exp(tX)g$ Next, we can define two smooth maps

$$s, s' \colon \mathbb{R} \to P \qquad s(t) = p\exp(tX) \qquad s'(t) = pg\exp(tY) = p\exp(tX)g$$

Then

$$D_p r_g(X^*_p) = (D_p r_g \circ D_0 s)(\partial) = D_0(r_g \circ s)(\partial) = D_0 s'(\partial) = Y^*_{pg} = (\mathrm{Ad}_{g^{-1}}(X))^*_{pg} \quad \square$$

Remark.

(1) The map
$$\mathfrak{g} \to \mathfrak{X}(P) \qquad X \mapsto X^*$$
is an injective homomorphism of Lie algebras, because G acts freely.

(2) For all $p \in P$, $\ker D_p\pi$ is spanned by the values of the fundamental vector fields at p: The map
$$\mathfrak{g} \to T_pP \qquad X \mapsto X_p^*$$
is a linear map of $\mathbb{R}$-vector spaces. This map is injective because X^* has no zeroes, and its image is in $\ker D_p\pi$. So by dimensional reasons, it is $\ker D_p\pi$.

Lemma. *A principal G–bundle P admits a global smooth section $s\colon M \to P$ if and only if it is isomorphic to the product bundle $M \times G \xrightarrow{\pi_1} M$.*

Proof. The product bundle has a smooth section
$$s\colon M \to M \times G \qquad m \mapsto (m, e)$$
If $f\colon P \to M \times G$ is an isomorphism, then $f^{-1} \circ s$ is a smooth section of P.

Conversely, suppose P admits a smooth section $s\colon M \to P$. Then define
$$f\colon M \times G \to P \qquad (m, g) \mapsto s(m)g$$
Clearly f is a smooth map. For any $h \in G$,
$$(f \circ r_h)(m, g) = f(m, gh) = s(m)gh = r_h(s(m)g) = (r_h \circ f)(m, g)$$
so f maps $\{m\} \times G$ to $\pi^{-1}(m)$. This map $f\colon \{m\} \times G \to \pi^{-1}(m)$ is bijective. To show injectivity, assume $s(m)g_1 = s(m)g_2$, then $g_1g_2^{-1} \in \mathrm{Stab}(s(m)) = \{e\}$, so $g_1 = g_2$. It is also surjective, since G acts transitively on $\pi^{-1}(m)$. So f^{-1}. One can check smoothness of f^{-1} in the local trivialization for p. □

Example.

(0) For any smooth manifold M and any Lie group G, the trivial bundle $M \times G \xrightarrow{\pi_1} M$ is a principal bundle.

(1) Let $H \subset G$ be a closed Lie subgroup. Then $P = G$ is a principal H–bundle over G/H with the action
$$G \times H \to G \qquad (g, h) \mapsto gh$$
The projection $\pi\colon G \to G/H$ admits local smooth sections. Let $U \subset G$ be open such that there is a smooth $s\colon U \to G$ with $\pi \circ s = \mathrm{id}_U$ and define
$$f\colon U \times H \to G \qquad (m, h) \mapsto s(m)h$$
This is a diffeomorphism between $U \times H$ and $\pi^{-1}(U)$ and for any $h' \in H$, we have
$$(r_{h'} \circ f)(m, h) = r_{h'}(s(m)h) = s(m)hh' = f(m, hh')$$
So all the requirements are satisfied such that G is a principal H–bundle.

(2) Let M be a smooth manifold and P the set of bases for tangent spaces of M. P has a C^∞ manifold structure such that π is smooth and P is the total space of a principal $\mathrm{GL}_n(\mathbb{R})$–bundle over M, $n = \dim(M)$.

Let $P \xrightarrow{\pi} M$ be a principal G–bundle and $M = \bigcup_\alpha U_\alpha$ a covering by local trivialisations. Suppose $U_\alpha \cap U_\beta \neq \varnothing$. Then the composition

$$(U_\alpha \cap U_\beta) \times G \xrightarrow{(\pi\times\varphi_\beta)^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\pi\times\varphi_\alpha} (U_\alpha \cap U_\beta) \times G$$

forms a diffeomorphism from $(U_\alpha \cap U_\beta) \times G$ to itself, which is the identity on the first factor. We denote this map by $(m, g) \mapsto (m, \overline{\psi_{\alpha\beta}}(m, g))$. By G–equivariance of the local trivializations, $\overline{\psi_{\alpha\beta}}(m, gh) = \overline{\psi_{\alpha\beta}}(m, g)h$ holds for any $m \in M$, $g, h \in G$, so we have $\overline{\psi_{\alpha\beta}}(m, g) = \overline{\psi_{\alpha\beta}}(m, e)g =: \psi_{\alpha\beta}(m)g$. These *transition maps* $\psi_{\alpha\beta}$ have the following properties:

(1) If $U_\alpha \cap U_\beta \neq \varnothing$, then $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ is a smooth map.

(2) $\psi_{\alpha\alpha}(m) = e$ for all $m \in U_\alpha$.

(3) $\psi_{\alpha\beta}(m) = \psi_{\beta\alpha}(m)^{-1}$ for all $m \in U_\alpha \cap U_\beta$.

(4) For all $m \in U_\alpha \cap U_\beta \cap U_\gamma$ we have the following:

$$\psi_{\alpha\beta}(m)\psi_{\beta\gamma}(m) = \psi_{\alpha\gamma}(m)$$

The properties (2) – (4) are summarized by saying that the maps $\psi_{\alpha\beta}$ satisfy the *cocycle conditions.* Property (3) follows directly from (2) and (4).

Now suppose we are given a smooth manifold M, an open covering $M = \bigcup_\alpha U_\alpha$ and smooth maps $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying the cocycle conditions. Then we can construct a principal G–bundle $P \xrightarrow{\pi} M$ trivial over each U_α such that $\psi_{\alpha\beta}$ are the transition maps of P:

$$P = \coprod_\alpha (U_\alpha \times G) \Big/ \sim$$

The equivalence relation $\sim$ is given as follows:

$$U_\alpha \times G \ni (m, g) \sim (m, \psi_{\alpha\beta}(m)g) \in U_\beta \times G \quad \iff \quad m \in U_\alpha \cap U_\beta$$

This really is an equivalence relation because $\psi_{\alpha\beta}$ satisfy the cocycle conditions. P is a smooth manifold that each $U_\alpha \times G$ projects to an open submanifold of P.

Now define a projection $\pi\colon P \to M$ by $\pi([(m, g)]) = m$. In the chart given by $U_\alpha \times G$ this is π_1 and so it is smooth. Also define an action

$$\mu\colon P \times G \to P \qquad ([(m, g)], h) \mapsto [(m, gh)]$$

It is well–defined and smooth. This definition of P, π, μ satisfies the properties (a) and (b) in the definition of a principal bundle. So we do indeed have a principal G–bundle defined from the $\psi_{\alpha\beta}$.

Example. Let M be a smooth manifold, (U_α, f_α) an atlas for M and $n = \dim M$. If $U_\alpha \cap U_\beta \neq \varnothing$, then we have the transition map

$$f_{\alpha\beta} \coloneqq f_\alpha \circ f_\beta^{-1} \colon f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$$

between open sets in $\mathbb{R}^n$. This $f_{\alpha\beta}$ is a diffeomorphism. Let $\psi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R})$ be defined as follows:

$$\psi_{\alpha\beta}(x) = D_{f_\beta(x)} f_{\alpha\beta} \in \mathrm{GL}_n(\mathbb{R})$$

The $\psi_{\alpha\beta}$ so defined are smooth and satisfy

$$\psi_{\alpha\alpha}(x) = D_{f_\alpha(x)} f_{\alpha\alpha} = D_{f_\alpha(x)} \mathrm{id}_{f_\alpha(U_\alpha)} = e \in \mathrm{GL}_n(\mathbb{R})$$

and, for any $x \in U_\alpha \cap U_\beta \cap U_\gamma$:

$$\psi_{\alpha\gamma}(x) = D_{f_\gamma(x)} f_{\alpha\gamma} = D_{f_\gamma(x)}(f_{\alpha\beta} \circ f_{\beta\gamma}) = D_{f_\beta(x)} f_{\alpha\beta} \circ D_{f_\gamma(x)} f_{\beta\gamma} = \psi_{\alpha\beta}(x)\psi_{\beta\gamma}(x)$$

We have checked that $\psi_{\alpha\beta}$ satisfy (2) and (4) of the cocycle conditions and (3) follows. Therefore $\psi_{\alpha\beta}$ define a principal $\mathrm{GL}_n(\mathbb{R})$–bundle over $\mathbb{R}^n$. This is the *bundle of bases/frames for tangent spaces* to M.

Definition. Let $P \xrightarrow{\pi} M$ and $P' \xrightarrow{\pi'} M'$ be principal G– resp. G'–bundles. A homomorphism f from P to P' is a pair of smooth maps

$$f' \colon P \to P' \qquad f'' \colon G \to G'$$

such that f'' is a homomorphism of Lie groups and

$$f'(pg) = f'(p)f''(g) \qquad \forall p \in P, g \in G$$

Notation. For a homomorphism f of principal bundles P and P', we usually denote both f' and f'' by f. We write

$$f \colon P \to P' \qquad f \colon G \to G'$$

and the equivariance is written

$$f(pg) = f(p)f(g)$$

Note that a homomorphism $P \to P'$ sends the fibers of P to the fibers of P'. There is a well–defined smooth $\overline{f} \colon M \to M'$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & P' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow[\ \overline{f}\ ]{} & M' \end{array}$$

Definition. Let $P \xrightarrow{\pi} M$ be a principal G–bundle and $H \subset G$ a Lie subgroup. A *reduction of the structure group* of P from G to H is an injective homomorphism f of a principal H–bundle $Q \to M$ into P such that $\overline{f} = \mathrm{id}_M$.

Example. Let $P = M \times G$ be the product bundle. Let $H = \{e\} \subset G$, $Q = M \times H$. Define

$$f\colon H \to G, e \mapsto e \qquad f\colon Q \to P, (m,e) \mapsto (m,e)$$

This f is a homomorphism and defines a reduction of the structure group of P to $\{e\}$.

Proposition. *Let $P \xrightarrow{\pi} M$ be a principal G–bundle and $H \subset G$ a Lie subgroup. The structure group of P can be reduced to H if and only if there is a system of local trivializations for P such that the corresponding transition maps $\psi_{\alpha\beta}$ take values in H.*

Proof. Assume there is a reduction $f\colon Q \to P$, where Q is a principal H-bundle. We may assume $f\colon H \to G$ is the inclusion. The corresponding transition maps take values in H. Conversely, suppose P admits local trivializations $U_\alpha \times G$ such that all transition maps $\psi_{\alpha\beta}$ take values in H. Then we can construct a principal H–bundle $Q \xrightarrow{\pi} M$ from the $\psi_{\alpha\beta}$. In each trivialization $U_\alpha \times G$ for P we have $U_\alpha \times H \subset U_\alpha \times G$. These inclusions induce an injective homomorphism $f\colon Q \to P$ giving a reduction of the structure group of P from G to H. □

2.1 Associated bundles

Definition. Let $P \xrightarrow{\pi} M$ be a principal G–bundle, F a smooth manifold and $\mu\colon G \times F \to F$ a left action of G on F. The *associated bundle* $E \xrightarrow{\pi_E} M$ is defined as follows:

$$E := P \times_G F := (P \times F)/\sim$$

where

$$(p,f) \sim (pg, g^{-1}f) \qquad \forall p \in P, f \in F, g \in G$$

and

$$\pi_E([(p,f)]) := \pi(p)$$

Let $U \subset M$ be an open set over which P is trivial and let

$$\pi^{-1}(U) \to U \times G \qquad p \mapsto (\pi(p), \varphi(p))$$

be a local trivialization. Performing the construction of E with $\pi^{-1}(U)$ in place of P, we obtain

$$\pi_E^{-1}(U) = (\pi^{-1}(U) \times F)/\sim \,\cong (U \times G \times F)/\simeq$$

with $\simeq$ defined as follows:

$$(m,h,f) \simeq (m, hg, g^{-1}f)$$

We claim that

$$(U \times G \times F)/\simeq \,\cong U \times F$$

To prove this, define $\psi_1\colon (U \times G \times F)/\simeq \;\to U \times F$ by $\psi_1([(m,h,f)]) = (m, hf)$ and $\psi_2\colon U \times F \to (U \times G \times F)/\simeq$ by $\psi_2(m,f) = [(m,e,f)]$. Then ψ_1 is well–defined and ψ_1, ψ_2 are mutually inverse:

$$\psi_2 \circ \psi_1([(m,h,f)]) = \psi_2(m,hf) = [(m,e,hf)] = [(m,h,f)]$$

$$\psi_1 \circ \psi_2(m,f) = \psi_1([(m,e,f)]) = (m,ef) = (m,f)$$

The associated bundle E has a unique differentiable structure in which the open subsets $\pi_E^{-1}(U)$ are open smooth submanifolds diffeomorphic to $U \times F$. This shows that E is a locally trivial smooth fiber bundle with fiber F and structure group G.

Example.

(1) If $\mu\colon G \times F \to F$ is the terminal action $(g,f) \mapsto f$ then $E = P \times_G F$ is diffeomorphic to $M \times F$ such that π_E corresponds to π_1.

(2) Let $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ be a homomorphism of Lie groups. Then G acts on $\mathbb{R}^n$ via ρ:

$$\mu\colon G \times \mathbb{R}^n \to \mathbb{R}^n \qquad (g,v) \mapsto \rho(g)v$$

In this case, $E = P \times_G \mathbb{R}^n =: P \times_\rho \mathbb{R}^n$ is a vector bundle over M.

(2') Suppose $V \to M$ is a vector bundle of rank k. The basis for fibers of V form a principal $GL_k(\mathbb{R})$–bundle $P \xrightarrow{\pi} M$. Take $\mathrm{id}\colon \mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}_k(\mathbb{R})$. Then $E = P \times_\rho \mathbb{R}^n$ is isomorphic to V.

(3) Let $P \xrightarrow{\pi} M$ be a principal G–bundle and $H \subset G$ a closed subgroup. Using the action

$$\mu\colon G \times G/H \to G/H \qquad (g, aH) \mapsto (ga)H$$

we can form the associated bundle E with fiber G/H.

Lemma. *In example (3), the associated bundle E with fiber G/H is diffeomorphic to the orbit space P/H, where H acts on P by restricting the G–action.*

Proof. We define to mutually inverse smooth maps ψ_1 and ψ_2 between E and P/H.

$$\psi_2\colon E \to P/H, [(p,aH)] \mapsto H(pa) \qquad \psi_1\colon P/H \to E, H(p) \mapsto [(p,H)]$$

These are indeed well–defined and smooth and $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1 = \mathrm{id}$. ☐

Proposition. *Let $P \xrightarrow{\pi} M$ be a principal G–bundle and $H \subset G$ a closed Lie subgroup. The structure group of P can be reduced to H if and only if the associated bundle E with fiber G/H has a section.*

Proof. Suppose the structure group of P can be reduced to H, so that there is a principal H–bundle $Q \to M$ and an injective homomorphism $f\colon Q \to P$. We claim that the composition $Q \xrightarrow{f} P \to P/H$ is constant on every fiber of Q. To see this, let $\alpha, \beta \in Q$

be in the same fiber of Q. Then there exists $h \in H$ such that $\alpha h = \beta$, so $f(\alpha)h = f(\beta)$ and thus $[f(\alpha)] = [f(\beta)]$, i.e. the images in P/H agree.

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d] & P \arrow[r] & P/H = E \\ M \arrow[urr] & & \end{tikzcd}$$

By the claim, this map factors through the projection $Q \to M$, and so gives a section $s\colon M \to E$. Conversely, suppose $E \xrightarrow{\pi_E} M$ admits a section $s\colon M \to E$. Define Q as the preimage of $s(M)$ under the map $P \to P/H = E$. The restriction to H of the G–action on P preserves $Q \subset P$ and is simply transitive on the fibers of $Q \to M$. Q is a principal H–bundle and the inclusion $Q \subset P$ is a reduction of the structure group of P to H. □

Definition. Suppose $P \xrightarrow{\pi} M$ is a principal G–bundle and $f\colon N \to M$ is a smooth map. Then define

$$f^*P \coloneqq \{(n,p) \in N \times P \mid f(n) = \pi(p)\}$$

$$\begin{tikzcd} f^*P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

f^*P is a principal G–bundle. It is called the *pullback bundle* obtained by pulling back $P \xrightarrow{\pi} M$ via f.

2.2 Connections

Let $P \xrightarrow{\pi} M$ be a principal G–bundle. Then π is a submersion and $D_p\pi\colon T_pP \to T_{\pi(p)}M$ has as kernel the tangent space at p to the fiber $\pi^{-1}(\pi(p))$. Moreover, $\ker(D_p\pi)$ is spanned by the fundamental vector fields X^* generated by the G–action on P. We call $\ker(D_p\pi) \eqqcolon V_p$ the *vertical tangent space* at p.

Definition. A *connection* on P is a choice of a complement H_p for V_p in T_pP for all $p \in P$ such that

(1) H_p depends smoothly on p.

(2) $D_p r_g(H_p) = H_{pg}$ for all $p \in P$, $g \in G$.

Remark. Property (1) is equivalent to saying that $\bigcup_{p\in P} H_p$ is a smooth subbundle H of TP. If $V = \bigcup_{p\in P} V_p$ is the vertical subbundle in TP with $V = \ker(D\pi)$, then H has to be a complement to V in TP, so that $TP = V \oplus H$. A connection H on P is a G–invariant smooth complement to V.

If H is a connection on P, then

$$D_p\pi\colon H_p \to T_{\pi(p)}M$$

is an isomorphism for all $p \in P$. Under this isomorphism, vector fields on M can be lifted to horizontal vector fields on P.

If $\alpha \in \Omega^1(P)$ is a 1–form on P, then at every point $p \in P$ with $\alpha_p \neq 0$, $\ker \alpha_p \subset T_pP$ is a hyperplane. If α is a 1–form with values in $\mathbb{R}^k$, then at every point $p \in P$ it defines a linear map

$$\alpha_p \colon T_pP \to \mathbb{R}^k$$

If α_p is surjective onto $\mathbb{R}^k$, then $\ker(\alpha_p) \subset T_pP$ is a subspace of codimension k.

Definition. Given a connection H on P, we define a 1–form ω on P with values in $\mathfrak{g}$ as follows:

$$\omega_p(X) = \begin{cases} 0 & \text{if } X \in H_p \\ A & \text{if } X = A^*_p \text{ for any } A \in \mathfrak{g} \end{cases}$$

where A^* is the fundamental vector field on P generated by the right action of $\exp(tA)$. Then ω is the *connection 1–form* corresponding to H.

At every point $p \in P$, we have $T_pP = V_p \oplus H_p$. H_p is in the kernel of ω_p and V_p contains only elements of the form A^*_p and for those $\omega(A^*_p) = A$. So ω is well–defined and $\ker \omega_p = H_p$, because $\omega_p \colon T_pP \to \mathfrak{g}$ is surjective. ω is also smooth, since H and A^* are smooth.

Lemma. *For $g \in G$, we have*

$$r_g^*\omega = \mathrm{Ad}_{g^{-1}} \omega$$

where $\mathrm{Ad}_{g^{-1}} \omega$ is the composition of $\omega \colon TP \to \mathfrak{g}$ and $\mathrm{Ad}_{g^{-1}} \colon \mathfrak{g} \to \mathfrak{g}$.

Proof. Let $X \in T_pP$. Since $(r_g^*\omega)_p(X) = \omega(D_p r_g(X))$, the claim of the lemma is equivalent to

$$\omega(Dr_g(X)) = \mathrm{Ad}_{g^{-1}}(\omega(X))$$

Both sides are linar in X, therefore it is enough to check the claim for $X \in V$ and $X \in H$.

If $X \in H_p$, then $\omega(Dr_g(X)) \in \omega(H_{pg}) = \{0\}$ by the G–invariance of H. Also $\mathrm{Ad}_{g^{-1}}(\omega(X)) = \mathrm{Ad}_{g^{-1}}(0) = 0$.

Now let $X \in V_p$. Since the fundamental vector fields span V, we can choose $Y \in \mathfrak{g}$ with $Y^*_p = X$. Then

$$\omega(Dr_g(X)) = \omega(Dr_g(Y^*_p)) = \omega((\mathrm{Ad}_{g^{-1}}(Y))^*_{pg}) = (\mathrm{Ad}_{g^{-1}}(Y))$$

and

$$\mathrm{Ad}_{g^{-1}}(\omega(X)) = \mathrm{Ad}_{g^{-1}}(\omega(Y^*_p)) = \mathrm{Ad}_{g^{-1}}(Y)$$ □

Proposition. *Suppose ω is a $\mathfrak{g}$–valued 1–form on P with the property that $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \omega$. Assume also that for fundamental vector fields A^*, we have $\omega(A^*) = A$. Then $H := \ker \omega$ is a connection on P.*

Proof. The two requirements on ω are consistent. If A^* is the fundamental vector field generated by A, then $Dr_g(A^*_p) = (\mathrm{Ad}_{g-1}(A))^*_{pg}$. This implies

$$(\mathrm{Ad}_{g^{-1}}(A))_{pg} = \omega((\mathrm{Ad}_{g^{-1}}(A))^*_{pg}) = \omega(Dr_g(A^*_p)) = r_g^*\omega(A^*_p)$$

For all $p \in P$, the map

$$\omega_p \colon T_pP \to \mathfrak{g}$$

is surjective, so $\ker \omega_p = H_p$ is a subspace of T_pP whose codimension is $\dim \mathfrak{g}$. The requirement $\omega(A^*) = A$ means that $\omega_p|_{V_p} \colon V_p \to \mathfrak{g}$ is an isomorphism for all $p \in P$. So H_p is a complement for V_p in T_pP.

Suppose $X \in H_p$. To prove G–invariance of H, we have to show $D_p r_g(X) \in H_{pg}$. Now $X \in H_p$ means that $\omega(X) = 0$. We have to prove $\omega(Dr_g(X)) = 0$. Indeed,

$$\omega(Dr_g(X)) = r_g^*\omega(X) = \mathrm{Ad}_{g^{-1}}(\omega(X)) = \mathrm{Ad}_{g^{-1}}(0) = 0 \qquad \square$$

Proposition. *Every principal G–bundle $P \xrightarrow{\pi} M$ admits a connection.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of M with the property that P restricted to each U_i is trivial:

$$\pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G$$

On the product bundle $U_i \times G$, there is a connection with

$$H_{(m,g)} = T_mU_i \times \{0\} \subset T_mU_i \oplus T_gG = T_{(m,g)}(U_i \times G)$$

Let ω_i be the connection 1–form on $\pi^{-1}(U_i)$, whose kernel corresponds to H under the trivialization $\pi^{-1}(U_i) \to U_i \times G$. Let $\{\rho_j\}_{j\in J}$ be a smooth partition of unity on M subordinate to the covering by the U_i, i.e. $\rho_j \colon M \to \mathbb{R}$ are smooth non-negative functions such that $\operatorname{supp}\rho_j$ are locally finite in M and $\sum_j \rho_j = 1$ and for all $j \in J$ there exists an $i \in I$ such that $\operatorname{supp}\rho_j \subset U_i$. Then define

$$\omega \coloneqq \sum_{j\in J} \pi^*\rho_j \cdot \omega_j = \sum_{j\in J} (\rho_j \circ \pi) \cdot \omega_j$$

where for all $j \in J$, $\omega_j \coloneqq \omega_i$ for some $i \in I$ such that $\operatorname{supp}\rho_j \subset U_i$, and the summands, which are supported only inside $\pi^{-1}(U_i)$ are being extended by 0 to all of P.

We claim that ω is a connection 1–form on P. To see this, we need to check that $r_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$ for all $g \in G$ and $\omega(A_p^*) = A_p$ for all $A \in \mathfrak{g}$, $p \in P$. Since $\pi^*\rho_j$ is constant under right G–action, the first equality follows by

$$\begin{aligned} r_g^*\omega = r_g^*\left(\sum_{j\in J}\pi^*\rho_j\cdot\omega_j\right) &= \sum_{j\in J}\pi^*\rho_j\cdot r_g^*\omega_j = \sum_{j\in J}\pi^*\rho_j\cdot \mathrm{Ad}_{g^{-1}}\,\omega_j = \\ &= \mathrm{Ad}_{g^{-1}}\left(\sum_{j\in J}\pi^*\rho_j\cdot\omega_j\right) = \mathrm{Ad}_{g^{-1}}\,\omega \end{aligned}$$

and the second by

$$\omega(A_p^*) = \left(\sum_{j\in J}\pi^*\rho_j\cdot\omega_j\right)(A_p^*) = \sum_{j\in J}\pi^*\rho_j(p)\cdot\omega_j(A_p^*) = \sum_{j\in J}\pi^*\rho_j(p)\cdot A_p = A_p \qquad \square$$

Proposition. *The set of connections on a principal G–bundle $P \xrightarrow{\pi} M$ is naturally an affine space whose vector space of translations is the space of 1–forms on M with values in the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g} \longrightarrow M$.*

More precisely, for any difference of connection 1–forms $\widetilde{\omega} = \omega_1 - \omega_2$ on P, there is a unique $\omega \in \Gamma((P \times_{\mathrm{Ad}} \mathfrak{g}) \otimes T^*M)$ such that

$$\pi^*\omega(Y) = [(p, \widetilde{\omega}(Y))] \qquad \forall Y \in T_pP$$

Proof. By the previous proposition, the set of connections P is non–empty, so we can choose a reference connection 1–form ω_0. For any connection 1–form ω_1, let $\widetilde{\omega} := \omega_1 - \omega_0$. For all $p \in P$, V_p is spanned by the values of the fundamental vector fields A^*_p with $A \in \mathfrak{g}$, but

$$\widetilde{\omega}(A^*_p) = \omega_1(A^*_p) - \omega_0(A^*_p) = A - A = 0$$

so $\widetilde{\omega}$ vanishes on V. Let $E := P \times_{\mathrm{Ad}} \mathfrak{g}$, then $\Omega^1(M, E) = \Gamma(T^*M \otimes E)$ is the vector space of 1–forms on M with values in the vector bundle E. We want to define $\omega \in \Omega^1(M, E)$ by

$$\omega(X) = [(p, \widetilde{\omega}(Y))]$$

for all $X \in T_mM$, where $Y \in T_pP$ is a lift of X at some $p \in \pi^{-1}(m)$, i.e. $D\pi(Y) = X$. We can always choose such a lift since $D\pi$ is surjective. For ω to be well–defined, we have to check that

$$[(p, \widetilde{\omega}(Y))] = [(q, \widetilde{\omega}(Z))]$$

for any lift $Z \in T_qP$ at $q \in \pi^{-1}(m)$. To see this, first let $q = p$ and $Z \in T_pP$ be a lift of X at p. Then

$$D\pi(Z - Y) = D\pi(Z) - D\pi(Y) = X - X = 0$$

so $Z - Y \in V$ and therefore

$$\widetilde{\omega}(Z) = \widetilde{\omega}(Y) - \widetilde{\omega}(Z - Y) = \widetilde{\omega}(Y)$$

Now let $q \in \pi^{-1}(m)$ be arbitrary and $Z \in T_qP$ a lift of X at q. There is a unique $g \in G$ such that $q = pg$. Since

$$D\pi(Dr_g(Y)) = D(\pi \circ r_g)(Y) = D\pi(Y) = X,$$

$Dr_g(Y) \in T_qP$ is a lift of X at q, so $\widetilde{\omega}(Z) = \widetilde{\omega}(Dr_g(Y))$. But then we have

$$[(q, \widetilde{\omega}(Z))] = [(q, \widetilde{\omega}(Dr_g(Y)))] = [(q, r_g^*\widetilde{\omega}(Y))] = [(pg, \mathrm{Ad}_{g^{-1}}\, \widetilde{\omega}(Y))] = [(p, \widetilde{\omega}(Y))].$$

This shows that although ω is not well–defined as an ordinary $\mathfrak{g}$–valued 1–form on M, it is well–defined as a 1–form on M with values in E.

Conversely, let $\omega \in \Omega^1(M, E)$. We have to check that with $\widetilde{\omega}$ defined by

$$\pi^*\omega(Y) = [(p, \widetilde{\omega}(Y))] \qquad \forall Y \in T_pP$$

the $\mathfrak{g}$–valued 1–form $\omega_1 = \omega_0 + \widetilde{\omega}$ is a connection 1–form. First, we show $\omega_1(X^*_p) = X$ for all $X \in \mathfrak{g}$ and $p \in P$: Since

$$[(p, \widetilde{\omega}(X^*_p))] = \pi^*\omega(X^*_p) = \omega(D\pi(X^*_p)) = \omega_{\pi(p)}(0) = \omega_{\pi(p)}(D_p\pi(0)) = (\pi^*\omega)_p(0) = [(p, 0)]$$

and the first element uniquely determines the representative of the equivalence class, we have

$$\omega_1(X_p^*) = \omega_0(X_p^*) + \widetilde{\omega}(X_p^*) = X + 0 = X.$$

Left to show is $r_g^*\omega_1 = \mathrm{Ad}_{g^{-1}}\,\omega_1$ for $g \in G$, but for $Y \in T_pP$,

$$\begin{aligned}[(pg, r_g^*\widetilde{\omega}(Y))] &= [(pg, \widetilde{\omega}(Dr_g(Y)))] = \pi^*\omega(Dr_g(Y)) = \omega(D\pi(Dr_g(Y))) = \\ &= \omega(D(\pi \circ r_g)(Y)) = \omega(D\pi(Y)) = \pi^*\omega(Y) = [(p, \widetilde{\omega}(Y))] = \\ &= [(pg, \mathrm{Ad}_{g^{-1}}\,\widetilde{\omega}(Y))],\end{aligned}$$

so $r_g^*\widetilde{\omega} = \mathrm{Ad}_{g^{-1}}\,\widetilde{\omega}$ and thus by linearity $r_g^*\omega_1 = \mathrm{Ad}_{g^{-1}}\,\omega_1$. $\square$

Let $P \xrightarrow{\pi} M$ be a principal G–bundle, ω a connection 1–form on P, $U_i, U_j \subset M$ open sets over which P is trivial. The trivializations

$$\psi_i \colon \pi^{-1}(U_i) \to U_i \times G$$

correspond to sections

$$s_i \colon U_i \to \pi^{-1}(U_i) \qquad m \mapsto \psi_i^{-1}(m, e)$$

$\omega_i := s_i^*\omega$ is a $\mathfrak{g}$–valued 1–form on $U_i \subset M$. Suppose $U_i \cap U_j \neq \varnothing$. Then on $U_i \cap U_j$ both ω_i and ω_j are defined. We have the smooth transition maps $\psi_{ij} \colon U_i \cap U_j \to G$ defined by

$$\psi_i \circ \psi_j^{-1} \colon (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G \qquad (m, g) \mapsto (m, \psi_{ij}(m)g)$$

and want to use them to find a formula for transition between ω_j and ω_i.

On G we have a canonical 1–form θ with values in $\mathfrak{g}$ defined by

$$\theta(A_g) = A \qquad \forall A \in \mathfrak{g}, g \in G$$

This is well–defined since it is equivalent to $\theta(X) = D\ell_{g^{-1}}(X)$ for $X \in T_gG$.

Lemma. *We have the following translation from ω_i to ω_j:*

$$\omega_j(X) = \mathrm{Ad}_{\psi_{ij}(m)}^{-1}\,\omega_i(X) + \psi_{ij}^*\theta(X) \qquad \forall X \in T_mM$$

Proof. Differentiating the function

$$s_j(m) = \psi_j^{-1}(m, e) = \psi_i^{-1} \circ \psi_i \circ \psi_j^{-1}(m, e) = \psi_i^{-1}(m, \psi_{ij}(m)) = s_i(m) \cdot \psi_{ij}(m)$$

gives for $X \in T_mM$

$$\begin{aligned}Ds_j(X) &= D(\mu \circ (s_i \times \psi_{ij}) \circ \Delta)(X) = (D\mu \circ D(s_i \times \psi_{ij}) \circ D\Delta)(X) = \\ &= D\mu(Ds_i(X), D\psi_{ij}(X)) = Dr_{\psi_{ij}(m)}(Ds_i(X)) + D\ell_{s_i(m)}(D\psi_{ij}(X))\end{aligned}$$

where we can transform the last summand using $A^*_{pg} = D\ell_p(A_g)$ like

$$\begin{aligned} D\ell_{s_i(m)}(D\psi_{ij}(X)) &= (D\ell_{s_i(m)} \circ D\ell_{\psi_{ij}(m)} \circ D\ell_{\psi_{ij}(m)^{-1}} \circ D\psi_{ij})(X) = \\ &= D\ell_{s_i(m)}((D\ell_{\psi_{ij}(m)^{-1}}(D\psi_{ij}(X)))_{\psi_{ij}(m)}) = \\ &= (D\ell_{\psi_{ij}(m)^{-1}}(D\psi_{ij}(X)))^*_{s_i(m)\psi_{ij}(m)} = \\ &= (\theta(D\psi_{ij}(X)))^*_{s_j(m)} = \\ &= (\psi^*_{ij}\theta(X))^*_{s_j(m)} \end{aligned}$$

Putting the above identities together we get the desired result

$$\begin{aligned} \omega_j(X) = \omega(Ds_j(X)) &= \omega(Dr_{\psi_{ij}(m)}(Ds_i(X)) + \omega((\psi^*_{ij}\theta(X))^*_{s_j(m)}) = \\ &= r^*_{\psi_{ij}(m)}\omega(Ds_i(X)) + \psi^*_{ij}\theta(X) = \operatorname{Ad}^{-1}_{\psi_{ij}(m)}\omega_i(X) + \psi^*_{ij}\theta(X) \end{aligned}$$ □

2.3 Parallel transport

Let $P \xrightarrow{\pi} M$ be a principal G–bundle, $H = \ker\omega$ a connection on P. For every $p \in P$, $D_p\pi|_H$ is an isomorphism $H_p \to T_{\pi(p)}M$. Every $X \in T_{\pi(p)}M$ has a unique preimage $X^* \in H_p$ under this isomorphism. Every vector field $X \in \mathfrak{X}(M)$ gives rise to a unique vector field X^* on P such that

$$X^*_p = (D_p\pi)^{-1}(X_{\pi(p)})$$

This construction has the following simple properties: $(fX)^* = \pi^*f \cdot X^*$ for any $f \in C^\infty(M,\mathbb{R})$ and $(X+Y)^* = X^* + Y^*$. But $[X,Y]^* \neq [X^*,Y^*]$: Although $[X,Y]_{\pi(p)} = D\pi([X^*,Y^*]_p)$ holds, $[X^*,Y^*]$ is not neccessarily horizontal. Denoting the projections from to the horizontal and vertical subbundles of TP by

$$\mathscr{V}\colon TP \to V \qquad \mathscr{H}\colon TP \to H$$

the above equality shows that

$$\mathscr{H}([X^*,Y^*]) = [X,Y]^*$$

Remark. In general $\mathscr{V}([X^*,Y^*]) \neq 0$, and is related to the curvature of H.

Definition. A smooth curve $c\colon [0,1] \to P$ is *horizontal* (wrt H) if $\dot{c}(t) \in H_{c(t)}$ for all $t \in [0,1]$.

Proposition. *Let $c\colon [0,1] \to M$ be a smooth curve and $p \in \pi^{-1}(c(0))$. Then there is a unique horizontal curve $\bar{c}\colon [0,1] \to P$ with $\bar{c}(0) = p$ and $\pi \circ \bar{c} = c$. $\bar{c}$ is called a* horizontal lift *of c.*

Proof. Given c and $p \in \pi^{-1}(c(0))$, there exists some smooth $\bar{c}\colon [0,1] \to P$ with $\bar{c}(0) = p$ and $\pi \circ \bar{c} = c$ (by local triviality). Any other lift of c to P with starting point p is of the form $\bar{c} \cdot g$ where $g\colon [0,1] \to G$ is a smooth map with $g(0) = e$. We need to find a g such that $\overline{c} \cdot g$ is horizontal. This is the case iff $\frac{\mathrm{d}}{\mathrm{d}t}(\overline{c}(t) \cdot g(t)) \in H$, i.e. $\omega(\frac{\mathrm{d}}{\mathrm{d}t}(\overline{c} \cdot g)) = 0$ for all $t \in [0,1]$. As in the lemma above, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{c} \cdot g) = D_t(\overline{c} \cdot g)(\partial) = (D\mu \circ D(\overline{c} \cdot g) \circ D_t\Delta)(\partial) = D_{\overline{c}} r_g(D_t\overline{c}(\partial)) + D_g\ell_{\overline{c}}(D_t g(\partial))$$

and

$$\begin{aligned} D_g\ell_{\overline{c}}(D_t g(\partial)) &= (D_g\ell_{\overline{c}} \circ D_e\ell_g \circ D_g\ell_{g^{-1}} \circ D_t g)(\partial) = D_g\ell_{\overline{c}}(((D_g\ell_{g^{-1}} \circ D_t g)(\partial))_g) = \\ &= ((D_g\ell_{g^{-1}} \circ D_t g)(\partial))^*_{\overline{c}\cdot g} = (D_g\ell_{g^{-1}}(\dot{g}))^*_{\overline{c}\cdot g} \end{aligned}$$

so

$$\begin{aligned} \omega\left(\frac{\mathrm{d}}{\mathrm{d}t}(\overline{c} \cdot g)\right) &= \omega(D_{\overline{c}} r_g(\dot{\overline{c}})) + D_g\ell_{g^{-1}}(\dot{g}) = r_g^*\omega(\dot{\overline{c}}) + D\ell_{g^{-1}}(\dot{g}) = \\ &= \mathrm{Ad}_{g^{-1}}\,\omega(\dot{\overline{c}}) + D\ell_{g^{-1}}(\dot{g}) = D\ell_{g^{-1}}(Dr_g(\omega(\dot{\overline{c}}))) + D\ell_{g^{-1}}(\dot{g}) = \\ &= (D\ell_{g^{-1}} \circ Dr_g)(\omega(\dot{\overline{c}}) + Dr_{g^{-1}}(\dot{g})) \end{aligned}$$

Since $D\ell_{g^{-1}} \circ Dr_g$ is an isomorphism, this is 0 if and only if

$$Dr_{g(t)^{-1}}(\dot{g}(t)) = -\omega(\dot{\overline{c}}(t))$$

So the statement of the proposition is just that this differential equation has a unique solution with $g(0) = e$. This is shows by the following lemma. □

Lemma. *Let* $X\colon [0,1] \to \mathfrak{g}$ *be smooth. There exists a unique* $g\colon [0,1] \to G$ *with* $g(0) = e$ *and*

$$Dr_{g(t)^{-1}}(\dot{g}(t)) = X(t)$$

Proof. On $G \times [0,1]$, X defines a time–independent vector field $\mathbb{X}$ with

$$\mathbb{X}_{(g,t)} = (X_g(t), \partial)$$

The flow of $\mathbb{X}$ is defined for all t (see the proof of completeness of left–invariant vector fields on G). Under the flow φ of $\mathbb{X}$ we have

$$\varphi_t(e,0) = (g(t), t)$$

for a $g(t)$ which solves our equation. This is the only solution with $g(0) = e$. □

Let $c\colon [0,1] \to M$ be a smooth curve. Define

$$P_c\colon \pi^{-1}(c(0)) \to \pi^{-1}(c(1)) \qquad p \mapsto \overline{c}(1)$$

where $\overline{c}$ is the unique horizontal lift of c with $\overline{c}(0) = p$.

The map P_c is the *parallel transport map* defined by c. P_c is invertible by running back along c. Except from change of direction, it is independent of the parametrization of c. The parallel transport map can also be defined for a piecewise smooth curve c by concatenating the P_{c_i} for c_i obtained by restricting c to subintervals of $[0,1]$ where it is smooth.

Fix a basepoint $m_0 \in M$. For every closed piecewise smooth curve $c\colon [0,1] \to M$ with $c(0) = c(1) = m_0$ we have

$$P_c\colon \pi^{-1}(m_0) \to \pi^{-1}(m_0)$$

Claim. *The set of all these P_c is a group with composition as group operation.*

Proof. $\{P_c\}$ has $\mathrm{id}_{\pi^{-1}(m_0)}$ as an element obtained as P_c for c the constant path at m_0. If c_1 and c_2 are two closed paths beginning and ending at m_0, then $P_{c_2} \circ P_{c_1} = P_{c_1 c_2}$, where $c_1 c_2$ denotes the concatenation of c_1 and c_2. This is associative and $P_c^{-1} = P_{\overline{c}}$ where $\overline{c}$ is c parametrized backwards. □

Fix a basepoint $p_0 \in P$ with $\pi(p_0) = m_0$. If P_c is one of the parallel transport maps defined above, then $P_c(p_0) \in \pi^{-1}(m_0)$. Since G acts simply transitively on the fiber, there exists a unique $g(c) \in G$ such that $P_c(p_0) = p_0 g(c)$.

Claim. *The map*

$$h\colon \{P_c\} \to G \qquad P_c \mapsto g(c)$$

is an injective homomorphism of groups.

Proof. Since $h(P_{c_1} \circ P_{c_2}) = h(P_{c_2 c_1}) = g(c_2 c_1)$ and $h(P_{c_1})h(P_{c_2}) = g(c_1)g(c_2)$ need to prove that $g(c_2 c_1) = g(c_1)g(c_2)$.

For any curve $c\colon [0,1] \to M$, the curve $\overline{c}$ from $p \in P$ to $P_c(p)$ is horizontal, since it is a horizontal lift of c. Since H is G–invariant, the G–action maps horizontal curves to horizontal curves, so for any $g \in G$, $\overline{c} \cdot g$ is a horizontal curve from pg to $P_c(p)g$. This means $P_c(pg) = P_c(p)g$, i.e. P_c is G–equivariant. So

$$p_0 g(c_2 c_1) = P_{c_2 c_1}(p_0) = P_{c_1} \circ P_{c_2}(p_0) = P_{c_1}(p_0 g(c_2)) = P_{c_1}(p_0) g(c_2) = p_0 g(c_1) g(c_2)$$

and thus $g(c_2 c_1) = g(c_1)g(c_2)$. We have proved that h is a homomorphism. Now suppose $P_c \in \ker h$, i.e. $g(c) = h(P_c) = e$. Then $P_c(p_0) = p_0$. Every $p \in \pi^{-1}(p_0)$ is of the form $p = p_0 g$ for some $g \in G$. But $P_c(p) = P_c(p_0 g) = P_c(p_0)g = p_0 g = p$ by the G–equivariance of P_c, so $P_c = \mathrm{id}_{\pi^{-1}(p_0)}$. Thus h is injective. □

Definition. The *holonomy group* $\mathrm{Hol}(H, p_0) = \mathrm{Hol}(p_0)$ of the connection H wrt $p_0 \in P$ is the subgroup of G obtained by parallel transport along closed loops based at p_0, i.e. $\mathrm{Hol}(H, p_0) = \mathrm{Im}\, h$.

The *restricted holonomy group* $\mathrm{Hol}_0(H, p_0) = \mathrm{Hol}_0(p_0)$ is the subgroup obtained by considering only parallel transports P_c for closed loops c which are contractible or null–homotopic.

Properties.

(1) $\mathrm{Hol}(H, p_1) = g^{-1} \mathrm{Hol}(H, p_0) g$ if $p_1 = p_0 g$.

(2) $\mathrm{Hol}(H, p_1) = \mathrm{Hol}(H, p_0)$ if p_1 is obtained from p_0 by parallel transport.

(3) If M is connected, $\mathrm{Hol}(H, p_0)$ and $\mathrm{Hol}(H, p_1)$ are conjugate in G for any $p_0, p_1 \in H$.

These properties also hold for Hol_0.

Proof.

(1) Consider $h_0 \in \mathrm{Hol}(H, p_0)$ and $h_1 \in \mathrm{Hol}(H, p_1)$ defined by $P_c(p_0) = p_0 h_0$ and $P_c(p_1) = p_1 h_1$ for the same curve c on M. Then

$$p_0 g h_1 = p_1 h_1 = P_c(p_1) = P_c(p_0 g) = P_c(p_0) g = p_0 h_0 g$$

so $g h_1 = h_0 g$ and $h_1 = g^{-1} h_0 g$.

(2) Let c be a smooth curve on M such that $p_1 = P_c(p_0)$ and let $h_1 \in \mathrm{Hol}(H, p_1) = \mathrm{Hol}(H, P_c(p_0))$. There exists a closed curve c_1 on M such that $P_{c_1}(P_c(p_0)) = P_c(p_0) h_1$. The curve $c c_1 \bar{c}$ is a closed curve in M based at $\pi(p_0)$, so there exists $h_0 \in \mathrm{Hol}(H, p_0)$ such that $P_{c c_1 \bar{c}}(p_0) = p_0 h_0$. But

$$P_c(p_0) h_1 = P_c \circ P_c^{-1} \circ P_{c_1} \circ P_c(p_0) = P_c(P_{c c_1 \bar{c}}(p_0)) = P_c(p_0 h_0) = P_c(p_0) h_0$$

so $h_1 = h_0$ and thus $h_1 \in \mathrm{Hol}(H, p_0)$. By using $\bar{c}$ instead of c, we get the other inclusion.

(3) Since M is connected, there is a smooth curve c from $\pi(p_0)$ to $\pi(p_1)$, which has a horizontal lift from p_0 to $P_c(p_0) \in \pi^{-1}(\pi(p_1))$, $\mathrm{Hol}(H, P_c(p_0)) = \mathrm{Hol}(H, p_0)$ by (2). By (1), this is conjugate to $\mathrm{Hol}(H, p_1)$. □

Theorem. *The restricted holonomy group* $\mathrm{Hol}_0(H, p_0)$ *is a connected Lie subgroup of* G*.*

Proof. By definition, $\mathrm{Hol}_0(H, p_0) \subset G$ is a subgroup. We claim that $\mathrm{Hol}_0(H, p_0)$ is connected, more precisely for any $g \in \mathrm{Hol}_0(H, p_0)$ there is a piecewise smooth curve $\widetilde{g}\colon [0,1] \to G$ with $\widetilde{g}(0) = e$, $\widetilde{g}(1) = g$ and $\widetilde{g}(s) \in \mathrm{Hol}_0(H, p_0)$ for all $s \in [0,1]$. The property $g \in \mathrm{Hol}_0(H, p_0)$ means that there is a piecewise smooth curve $c\colon [0,1] \to M$ with $c(0) = c(1) = m_0$ such that $P_c(p_0) = p_0 g$ and c is contractible as a curve based at m_0. There exists a piecewise smooth map

$$H\colon [0,1] \times [0,1] \to M$$

such that

$$H(t,0) = m_0, \quad H(t,1) = c(t), \quad H(0,s) = H(1,s) = m_0 \qquad \forall s,t \in [0,1]$$

For every $s \in [0,1]$,

$$c_s\colon [0,1] \to M \qquad t \mapsto H(t,s)$$

is a piecewise smooth curve in M based at m_0. Define $\widetilde{g}$ by $p_0 \widetilde{g}(s) = P_{c_s}(p_0)$. This is piecewise smooth in s. We have $\widetilde{g}(0) = e$ since c_0 is constant and $\widetilde{g}(1) = g$ since $c_1 = c$. $\widetilde{g}(s) \in \mathrm{Hol}_0(H, p_0)$ because each c_s is a closed loop based at m_0 and is contractible. The proof of the theorem is completed by the following proposition. □

Proposition. *Let* G *be a Lie group and* $H \subset G$ *a subgroup with the property that every* $g \in H$ *can be connected to* e *by a piecewise smooth curve in* H*. Then* H *is a Lie subgroup of* G*.*

Proof. Let

$$\mathfrak{h} = \{\dot{c}(0) \mid c\colon [0,1] \to G \text{ piecewise smooth}, \ c(0) = e, \ \forall t\colon c(t) \in H\}.$$

We claim that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g} = L(G)$. If $X \in \mathfrak{g}$ is in $\mathfrak{h}$, then so is λX by reparametrizing c with $\dot{c}(0) = E$. If c_1 and c_2 are two curves with $\dot{c_1}(0) = X \in \mathfrak{h}$ and $\dot{c_2} = Y \in \mathfrak{h}$, then, with

$$c\colon [0,1] \to G, \quad t \mapsto c_1(t)c_2(t),$$

we have $\dot{c}(0) = X+Y$, so $\mathfrak{h} \in \mathfrak{g}$ is a linear subspace. It remains to prove that for $X, Y \in \mathfrak{h}$ we have $[X,Y] \in \mathfrak{h}$. Let c_1, c_2 be as above. Consider $c(t^2) = c_1(t)c_2(t)c_1(-t)c_2(-t)$. Without loss of generalit, we may take c_1, c_2 defined on $(-\varepsilon, 1]$ for some small $\varepsilon > 0$. The defining formula for c makes sense for small positive t. We have $\dot{c}(0) = [X,Y]$ and c satisfies $c(t) \in H$ for all t, so $[X,Y] \in \mathfrak{h}$.

Now we define a subbundle $E \subset TG$ by $E_g \coloneqq D\ell_g(\mathfrak{h}) \subset T_gG$ forall $g \in G$. We prove that E is integrable: Since $\mathfrak{h}$ is closed under $[-,-]$, E is involutive. By the Frobenius theorem, E is integrable, i.e. there exists an integral submanifold through every point. Let $K \subset G$ be the maximal connected integral submanifold of E through $e \in G$. K is a connected Lie subgroup with Lie algebra $\mathfrak{h}$.

Next, we show that $H \subset K$. By assumption, every $h \in H$ is connected to $e \in G$ by a piecewise smooth $c\colon [0,1] \to G$ with $c(t) \in H$ for all t. Suppose c is smooth. For a fixed t_0 define $\overline{c}(t) = c(t_0)^{-1}c(t) \in H$. We have $\overline{c}(t_0) = e$ and $\dot{\overline{c}}(t_0) = D_{c(t_0)}\ell_{c(t_0)^{-1}}(\dot{c}(t_0)) \in \mathfrak{h}$, so $\dot{c}(t_0) = D_e\ell_{c(t_0)}(\dot{\overline{c}}(t_0)) \in E_{c(t_0)}$ for all t_0. Hence c is contained in K and in particular $h = c(1) \in K$. For piecewise smooth c we repeat this argument for each subinterval of $[0,1]$ where c is smooth. So $H \subset K$.

To prove that $K \subset H$, pick a basis $X_1, \dots, X_k$ for $\mathfrak{h}$. For each i we can choose a curve c_i with $c_i(0) = e$, $c_i(t) \in H \, \forall t$ and $\dot{c}_i(0) = X_i$. Define

$$f\colon (-\varepsilon, \varepsilon)^k \to G, \qquad (t_1, \dots, t_k) \mapsto c_1(t_1)c_2(t_2)\cdots c_k(t_k).$$

Since each c_i is contained in H and $H \subset G$ is a subgroup, $f(t_1, \dots, t_k) \in H$ for all t_i and $f(0, \dots, 0) = e$. The linear map $D_{(0,\dots,0)}f$ is an isomorphism between $\mathbb{R}^k$ and $\mathfrak{h}$. So, for $\varepsilon > 0$ small enough, f is an immersion. We obtain an immersed submanifold through e which is an integral submanifold for E, since $\operatorname{Im}(Df)$ is contained in E at every point. The maximal integral submanifold of E through e is K, so $\operatorname{Im} f \subset K$. In fact, $\operatorname{Im} f$ and K have the same dimension, so $\operatorname{Im} f$ is an open neighbourhood of e in K. Since $\operatorname{Im} f \subset H$ by definition of f, $K \subset H$ is true locally near e. Since K is a connected Lie subgroup, each element of K is a product of finitely many elements of $\operatorname{Im} f \subset H$. since H is also a subgroup, $H \subset K$. □

2.4 Curvature

Definition. The curvature of H is the following $\mathfrak{g}$–valued 2–form on P:

$$\Omega(X,Y) = \mathrm{d}\omega(\mathscr{H}X, \mathscr{H}Y) \qquad \forall X, Y \in T_pP, p \in P$$

For $X, Y \in T_pP$, $p \in P$, we have

$$\begin{aligned}(r_g^*\Omega)(X,Y) &= \Omega(Dr_g(X), Dr_g(Y)) = \mathrm{d}\omega(\mathscr{H}Dr_g(X), \mathscr{H}Dr_g(Y)) = \\ &= \mathrm{d}\omega(Dr_g(\mathscr{H}X), Dr_g(\mathscr{H}Y)) = (r_g^*\mathrm{d}\omega)(\mathscr{H}X, \mathscr{H}Y) = \\ &= \mathrm{d}(r_g^*\omega)(\mathscr{H}X, \mathscr{H}Y) = (\mathrm{d}\,\mathrm{Ad}_{g^{-1}}\,\omega)(\mathscr{H}X, \mathscr{H}Y) = \\ &= \mathrm{Ad}_{g^{-1}}\,\mathrm{d}\omega(\mathscr{H}X, \mathscr{H}Y) = \mathrm{Ad}_{g^{-1}}\,\Omega(X,Y),\end{aligned}$$

so $r_g^*\Omega = \mathrm{Ad}_{g^{-1}}\,\Omega$.

Proposition (Structure equation)**.** $\Omega(X,Y) = \mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)]$.

Proof. Both sides of the claim are bilinear and skew–symmetric, so we may assume that each X, Y is either in V or in H. In the case $X, Y \in H$, we have $\omega(X) = \omega(Y) = 0$, and $\Omega(X,Y) = \mathrm{d}\omega(X,Y)$ by definition of Ω. If instead $X, Y \in V_p$ with $p \in P$, we may assume $X = A_p^*$ and $Y = B_p^*$ for $A, B \in \mathfrak{g}$. Since $\mathscr{H}X = \mathscr{H}Y = 0$, we have $\Omega(X,Y) = 0$. The structure equation then holds since

$$\begin{aligned}\mathrm{d}\omega(A^*, B^*) &= A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega([A^*, B^*]) = \\ &= A^*(B) - B^*(A) - \omega([A,B]^*) = 0 - 0 - [A,B] = \\ &= -[\omega(A^*), \omega(B^*)]\end{aligned}$$

so in particular $\mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)] = 0 = \Omega(X,Y)$. In the third case, $X_p \in H_p$ and $Y_p \in V$, where X is a vector field and $Y_p = B_p^*$ for some $B \in \mathfrak{g}$. Again,

$$\Omega(X,Y) = \mathrm{d}\omega(\mathscr{H}X, \mathscr{H}Y) = \mathrm{d}\omega(X, 0) = 0.$$

Then we have

$$\begin{aligned}\mathrm{d}\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = \\ &= X(B) - 0 - \omega([X, B^*]) = -\omega([X, B^*]) = 0\end{aligned}$$

by the following lemma. Since also $[\omega(X), \omega(Y)] = [0, B] = 0$, this completes the proof of case 3 and of the proposition. □

Lemma. *Let X be a vector field with $X_p \in H_p$. Then $[X, B_p^*] \in H_p$.*

Proof. For any two vector fields X and Y we have

$$[X,Y] = L_XY = -\frac{\mathrm{d}}{\mathrm{d}t} D\varphi_t(Y)\Big|_{t=0}$$

With this we have for X horizontal and $Y = B_p^*$:

$$[X, B_p^*] = -[B_p^*, X] = \frac{\mathrm{d}}{\mathrm{d}t} Dr_{\exp(tB)}(X)\Big|_{t=0} \in H$$

since H is invariant under right G–action. □

Definition. Let α be a differential k–form on P. Then D is defined by

$$D\alpha = \mathrm{d}\alpha \circ \mathscr{H}, \quad \text{i.e. } D\alpha(X_1, \ldots, X_{k+1}) = \mathrm{d}\alpha(\mathscr{H}X_1, \ldots, \mathscr{H}X_{k+1}).$$

D depends on the connection H and is called the *covariant derivative* defined by H.

Proposition (Bianchi identity)**.** *$D\Omega = 0$, in other words $\mathrm{d}\Omega(\mathscr{H}X, \mathscr{H}Y, \mathscr{H}Z) = 0$ for all $X, Y, Z \in T_pP$.*

Proof. Let $X, Y, Z \in H_p$, then

$$\begin{aligned}
\mathrm{d}\Omega(X, Y, Z) &= L_X(\Omega(Y, Z)) + L_Y(\Omega(Z, X)) + L_Z(\Omega(X, Y)) \\
&\quad - \Omega([X, Y], Z) - \Omega([Z, X], Y) - \Omega([Y, Z], X) \\
&= L_X(\mathrm{d}\omega(Y, Z)) + L_Y(\mathrm{d}\omega(Z, X)) + L_Z(\mathrm{d}\omega(X, Y) \\
&\quad - \mathrm{d}\omega([X, Y], Z) - \mathrm{d}\omega([Z, X], Y) - \mathrm{d}\omega([Y, Z], X) \\
&= \mathrm{d}\mathrm{d}\omega(X, Y, Z) = 0
\end{aligned}$$

□

Proposition. *Let $P \xrightarrow{\pi} M$ be a principal G–bundle an H a connection of P with curvature Ω. The following conditions are equivalent:*

(1) $\Omega = 0$.

(2) H is involutive, i.e. closed under $[-,-]$.

(3) H is integrable.

Proof. Conditions (2) and (3) are equivalent by the Frobenius theorem. Assume $X, Y \in H$. Then

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]),$$

so $\Omega = 0$ iff $[X, Y] \in H$ whenever $X, Y \in H$. This shows the equivalence of (1) and (2). □

Let $L \subset P$ be the maximal connected integral submanifold for H with $p_0 \in L$. If c is any loop in M based at $m_0 = \pi(p_0)$, then the horizontal lift $\overline{c}$ of c with initial value p_0 is contained in L. Note that L is a covering space of M.

$$\pi|_L \colon L \to M \qquad D(\pi|_L) \colon T_pL = H_p \to T_{\pi(p)}M$$

$D(\pi|_L)$ is an isomorphism, so $\pi|_L$ is a local diffeomorphism.

Definition. The connection $H = \ker \omega$ is *flat* if $\Omega = 0$.

Now let H be arbitrary, not neccessarily flat. For $p_0 \in P$ let

$$H(p_0) = \{p \in P \mid \ p \text{ is obtained from } p_0 \text{ by parallel transport}\}$$

Note that $\mathrm{Hol}_0(H, p_0)$ is a connected Lie subgroup of G. Moreover, $\mathrm{Hol}_0(H, p_0)$ is the connected compontent of e in $\mathrm{Hol}(H, p_0)$. $\mathrm{Hol}(H, p_0)$ has at most countable many connected components, each diffeomorphis to $\mathrm{Hol}_0(H, p_0)$. So $\mathrm{Hol}(H, p_0) \subset G$ is a Lie subgroup as well.

Proposition. *If M is connected, then $H(p_0)$ is a principal* $\mathrm{Hol}(H, p_0)$*–bundle.*

Proof. $H(p_0)$ is a subset of P, so we have a projection $\pi\colon H(p_0) \to M$ by restricting $\pi\colon P \to M$. Consider

$$H(p_0) \cap \pi^{-1}(m_0) = \{p_0 g(c) \mid c \text{ is a closed loop based at } m_0\}.$$

Since $g(c) \in \mathrm{Hol}(H, p_0)$, this shows that $\mathrm{Hol}(H, p_0)$ acts simply transitively on the fiber of $H(p_0)$ over m_0. G acts on P on the right, and we restrict this action to the subgroup $\mathrm{Hol}(H, p_0) \in G$.

We claim that the restricted action maps $H(p_0)$ to itself. To prove this, let $p \in H(p_0)$ and $\overline{c}$ be a horizontal curve connecting p_0 to p. Assume $g \in \mathrm{Hol}(H, p_0)$. Then $r_g \circ \overline{c}$ is a horizontal curve connecting $p_0 g$ to pg. Because $g \in \mathrm{Hol}(H, p_0)$, there is a horizontal curve from p_0 to $p_0 g$. Concatenating with $r_g \circ \overline{c}$ gives a piecewise smooth horizontal curve from p_0 to pg, so $pg \in H(p_0)$.

Next, we want to show that the action of $\mathrm{Hol}(H, p_0)$ on $H(p_0)$ is simply transitive on every fiber of $H(p_0) \xrightarrow{\pi} M$. Let $p \in H(p_0)$ and $\overline{c}$ be a horizontal curve from p_0 to p. For every $g \in \mathrm{Hol}(H, p_0)$, $pg \in H(p_0)$ by the previous paragraph. Suppose $p' \in H(p_0) \cap \pi^{-1}(\pi(p))$. There exists a horizontal curve $\overline{c}'$ from p_0 to p'. Since both p and p' are connected to p_0 by horizontal curves, there is a horizontal curve from p to p'. So there exists $g' \in \mathrm{Hol}(H, p)$ such that $p' = pg'$. Since there is a horizontal curve between p_0 and p, we have $\mathrm{Hol}(H, p) = \mathrm{Hol}(H, p_0)$. We have shown that $\mathrm{Hol}(H, p_0)$ acts transitively on $H(p_0) \cap \pi^{-1}(\pi(p))$ for all $p \in H(p_0)$. Since the action of G on P has trivial stabilizers, the restricted action of $\mathrm{Hol}(H, p_0)$ on $H(p_0)$ has trivial stabilizers.

By connectedness of M, $H(p_0)$ intersects every fiber of P. Take local trivializations for P:

$$\psi\colon \pi^{-1}(U) \to U \times G$$

Since $\mathrm{Hol}_0(H, p_0) \subset G$ is a Lie subgroup, $U \times \mathrm{Hol}_0(H, p_0)$ is a smooth manifold and so is $U \times \mathrm{Hol}(H, p_0)$. Using the restriction of ψ to $\pi^{-1}(U) \cap H(p_0)$, we can identify $\pi^{-1}(U) \cap H(p_0)$ with $U \times \mathrm{Hol}(H, p_0)$. $H(p_0)$ has a unique smooth structure for which these identifications are diffeomorphisms. With respect to this smooth structure on $H(p_0)$, the action of $\mathrm{Hol}(H, p_0)$ on $H(p_0)$ discussed before is smooth. ☐

Definition. Let $p_0 \in P$ and H be a connection on P. The principal $\mathrm{Hol}(H, p_0)$–bundle $H(p_0)$ is the *holonomy bundle* of H through p_0.

Remark. Consider $H(p_0) \hookrightarrow P$. This is a homomorphism of principal bundles where the corresponding homomorphism of Lie groups is $\mathrm{Hol}(H, p_0) \hookrightarrow G$.

A reduction of the structure group of P to H is a principal H–bundle $Q \to M$ together with a homomorphism $Q \to P$ with respect to the inclusion $H \hookrightarrow G$ such that

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & \swarrow & \\ M & & \end{array}$$

commutes.

The existence of such a reduction is equivalent to a section of the bundle

$$\begin{array}{ccc} G/H & \longrightarrow & P/H \\ & & \downarrow \\ & & H \end{array}$$

Extending this definition slightly, we have proved that if H is a connection of P and $Q := H(p_0)$ is the holonomy bundle through a basepoint $p_0 \in P$, then $Q \hookrightarrow P$ defines a reduction of the structure group of P to $\mathrm{Hol}(H, p_0) \subset G$.

Suppose Q is a reduction of the structure group G of P to a subgroup $G' \subset G$. Let H be a connection on Q.

Claim. *H extends uniquely to P.*

Proof. For $p \in P$ consider some $p' \in Q \cap \pi^{-1}(\pi(p))$. The connection H on Q defines $H_{p'} \subset T_{p'}Q$. $H_{p'}$ is a complement to the vertical subspace at p' in Q and also in P. There exists $g \in g$ such that $p'g = p$. Define $H_p := Dr_g(H_{p'})$. This is a horizontal subspace at p in P.

This defines a connection on P! H is clearly horizontal. It is also smooth, so we only have to check G–invariance. Let $\overline{p} \in \pi^{-1}(\pi(p))$. There exists $\overline{g} \in G$ such that $\overline{p} = p\overline{g}$. We have to check that

$$Dr_{\overline{g}}(H_p) = H_{\overline{p}}$$

There exist $p' \in Q \cap \pi^{-1}(\pi(p))$ and $g' \in G$ such that $p' = pg'$. Then $p = p'g'^{-1}$ and $\overline{p} = p\overline{g} = p'g'^{-1}\overline{g}$, so

$$H_{\overline{p}} = Dr_{g'^{-1}\overline{g}}(H_{p'}) = Dr_{\overline{g}}(Dr_{g'^{-1}}(H_{p'})) = Dr_{\overline{g}}(H_p).$$

Suppose $\overline{H}$ is a connection on P which restricts to the given H on Q, i.e. $\overline{H_p} = H_p$ if $p \in Q$. Then $\overline{H}$ is the connection defined above, because every fiber of P contains a point in Q and the connection $\overline{H}$ on P is completely determined by G–invariance and $\overline{H}$ at a single point in every fiber. □

Definition. Let $\overline{H}$ be a connection on a principal G–bundle $P \xrightarrow{\pi} M$ and $G' \subset G$ a Lie subgroup. $\overline{H}$ is *reducible to the subgroup G'* if there is a reduction $Q \to M$ of P to a principal G'–bundle and a connection H on Q whose extension to P is $\overline{H}$.

Proposition. *Every connection $\overline{H}$ on P is reducible to* $\mathrm{Hol}(\overline{H}, p_0)$.

Proof. Set $Q = H(p_0)$ and H the restriction of $\overline{H}$ to Q. □

Theorem (Ambrose–Singer)**.** *Le $P \xrightarrow{\pi} M$ be a principal G–bundle with a connection $H = \ker \omega$ and curvature Ω, where M is connected, and let $p \in P$. Define*

$$\mathfrak{g}' := \{\Omega(X_p, Y_p) \mid X_p, Y_p \in T_pP\} \subset \mathfrak{g}$$

Then $\mathfrak{g}'$ is the Lie algebra of $\mathrm{Hol}(H, p)$.

Proof. We replace P by the hononomy bundle $Q = H(p)$ through P. We work on Q whose structure group is $G' = \mathrm{Hol}(H, p)$. We have to prove that the values of Ω span the full Lie algebra of G'.

Take a basis $A_1, \ldots, A_k$ for the subspace $\mathfrak{g}'$. The A_i induce fundamental vector fields A_i^* on Q. Let $X_1, \ldots, X_n$ be a basis for H_p and $X_1^*, \ldots, X_n^*$ extensions of the X_i to horizontal vector fields. Let $S \subset T_pQ$ be the span of the $(A_i^*)_p$ and $(X_j^*)_p = X_j$. We extend S to a smooth distribution on Q by setting

$$S_q \coloneqq \mathrm{span}\{(A_1^*)_q, \ldots, (A_k^*)_q\} \oplus H_q.$$

We claim that S is integrable as a distribution on Q. To check that S is closed under $[-,-]$, we can check at p: First, $[A_i^*, A_j^*] = [A_i, A_j]^*$, so we need to check that $\mathfrak{g}' = \mathrm{span}\{A_1, \ldots, A_k\}$ is a Lie subalgebra. Second, $[A_i^*, X_j^*]$ is horizontal by a previous Lemma, so it is in $H \subset S$. And third,

$$[X_i^*, X_j^*] = \mathscr{H}([X_i^*, X_j^*]) + \mathscr{V}([X_i^*, X_j^*]).$$

Since the horizontal part is in S, we only need to prove that $\mathscr{V}([X_i^*, X_j^*]) \in S$. Since the fundamental vector fields span the vertical subspace, we can write

$$\mathscr{V}([X_i^*, X_j^*](p)) = B_p^* \quad \text{for some } B \in \mathfrak{g},$$

so $\omega(\mathscr{V}([X_i^*, X_j^*](p))) = B$. Then

$$\begin{aligned}\Omega(X_i^*(p), X_j^*(p)) &= \mathrm{d}\omega(X_i^*(p), X_j^*(p)) = L_{X_i^*}(\omega(X_j^*)) - L_{X_j^*}(\omega(X_i^*)) - \omega([X_i^*, X_j^*]) = \\ &= -\omega([X_i^*, X_j^*](p)) = -\omega(\mathscr{V}([X_i^*, X_j^*](p))) = -B\end{aligned}$$

so $B \in \mathrm{span}\{A_1, \ldots, A_k\}$. The equation $\mathscr{V}([X_i^*, X_j^*](p)) = B_p^*$ then shows that $\mathscr{V}([X_i^*, X_j^*](p)) \in \mathrm{span}\{(A_1^*)_p, \ldots, (A_k^*)_p\} \subset S$.

We have proved that S is closed under $[-,-]$, so it is integral by the Frobenius theorem.

Let L be the maximal connected integral submanifold of S through the point $p \in Q$. Every point in Q can be reached from p by parallel transport of p. The correspoinding horizontal curve is tangent to H and therefore tangent to S. Therefore the whole curve is contained in L and so $L = Q$. Since

$$k + \dim M = \mathrm{rank}\, S = \dim L = \dim Q = \dim G' + \dim M,$$

we have $\dim \mathfrak{g}' = k = \dim G' = \dim L(G')$, so $\mathfrak{g}' = L(G')$. □

By definition, H is G–invariant. So, since $r_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$, we also have $r_g^*\Omega = \mathrm{Ad}_{g^{-1}}\,\Omega$, i.e. $\Omega(Dr_g(X), Dr_g(Y)) = \mathrm{Ad}_{g^{-1}}\,\Omega(X, Y)$ for all $X, Y \in T_pP$. Since $\mathrm{Ad}_{g^{-1}}$ is a linear isomorphism, there is as much curvature at pg as there is at p for all $g \in G$.

With $E = P \times_{\mathrm{Ad}} \mathfrak{g} = P \times \mathfrak{g}/\simeq$ where $(p, A) \simeq (pg, \mathrm{Ad}_{g^{-1}} A)$ for all $g \in G$, the bundle $\pi_E \colon E \to M, [(p, A)] \to \pi(p)$ is a vector bundle on M whose fiber at every point $m \in M$ is isomorphic to $\mathfrak{g}$. We interpret the curvature Ω of a connection H on P as a 2–form on M with values in E, i.e.

$$\Omega \in \Omega^2(M) \otimes E = \Gamma(\Lambda^2 T^*M \otimes E)$$

For $X, Y \in T_m M$ we should have $\Omega(X,Y) \in E_m = \pi_E^{-1}(m)$. Given $X, Y \in T_m M$, choose preimages $\widetilde{X}, \widetilde{Y} \in T_p P$ with $\pi(p) = m$ and $D_p\pi(\widetilde{X}) = X$, $D_p\pi(\widetilde{Y}) = Y$. Then

$$\Omega(X,Y) = [(p, \Omega(\widetilde{X}, \widetilde{Y}))] \in E_{\pi(p)} = E_m$$

If we replace $\widetilde{X}$ by $\widetilde{X}' \in T_p P$ with $D_p\pi(\widetilde{X}') = X$, then $\widetilde{X} - \widetilde{X}' \in \ker D_p\pi = V_p$, so

$$\Omega(\widetilde{X}', \widetilde{Y}) = \Omega(\widetilde{X}' - \widetilde{X}, \widetilde{Y}) + \Omega(\widetilde{X}, \widetilde{Y}) = \Omega(\widetilde{X}, \widetilde{Y})$$

This shows that $\Omega(X,Y)$ is independent of the choice of $\widetilde{X}$ at p and similarily for $\widetilde{Y}$ at p.

Any $p' \in P$ with $\pi(p') = m$ is of the form $p' = pg$ for some $g \in G$. Then

$$(p, \Omega(\widetilde{X}, \widetilde{Y})) \simeq (pg, \mathrm{Ad}_{g^{-1}} \Omega(\widetilde{X}, \widetilde{Y})) = (p', r_g^* \Omega(\widetilde{X}, \widetilde{Y})) = (p', \Omega(Dr_g(\widetilde{X}), Dr_g(\widetilde{Y})))$$

and $D\pi(Dr_g(\widetilde{X})) = X$, $D\pi(Dr_g(\widetilde{Y})) = Y$. This shows that $\Omega(X,Y) := [(p, \Omega(\widetilde{X}, \widetilde{Y}))]$ is well–defined as an element of E_m.

2.5 Global gauge transformations

Definition. An *automorphism* (or *global gauge transformation*) of P is a diffeomorphism $\phi\colon P \to P$ such that $\pi \circ \phi = \pi$ ad $\phi(pg) = \phi(p)g$ for all $g \in G$.

Every such ϕ has an inverse ϕ^{-1} which is a diffeomorphism. Moreover, $\pi \circ \phi^{-1} = \pi$ and $\phi^{-1}(pg) = \phi^{-1}(p)g$ for all $g \in G$. So the automorphisms of P form a group.

Definition. The group of automorphisms of P is called the *gauge group* $\mathscr{G}$ of P.

Proposition. *$\mathscr{G}$ is the space of sections of the bundle $F \to M$ with fiber G associated to P by the conjugation action of G on itself.*

Proof. F is defined by $F = P \times G/\simeq$ where $(p,h) \simeq (pg, g^{-1}hg)$ for all $g \in G$. Let $\phi \in \mathscr{G}$. Then $\phi(p) = pu(p)$ for some smooth $u\colon P \to G$. The definition of automorphisms gives

$$(pg)u(pg) = pu(p)g \Rightarrow gu(pg) = u(p)g \Rightarrow u(pg) = g^{-1}u(p)g$$

Define a section $s\colon M \to F$ by $s(m) = [(p, u(p))]$ for any $p \in \pi^{-1}(m)$. Since

$$(p, u(p)) \simeq (pg, g^{-1}u(p)g) = (pg, u(pg)),$$

s is well–defined and so it is a smooth section of F.

Conversely, suppose $s\colon M \to F$ is a smooth section of F. Define $u\colon P \to G$ by $p \mapsto g$ if $[(p,g)] = s(\pi(p))$. Then $\phi(p) = pu(p)$ is a gauge transformation of P. □

The gauge group $\mathscr{G}$ acts on connections. If ω is a connection 1–from defining a connection H on P, then $\phi^*\omega$ is also a connection 1–form defining the pulled–back connection:

$$(\phi^* H)_p = (D_p\phi)^{-1} H_{\phi(p)}$$

For every H we can think of its curvature Ω as a section of $\Lambda^2 T^*M \otimes E$, equivalently a 2–form on M with values in E, where $E = P \times_{\mathrm{Ad}} \mathfrak{g}$. The action of $\mathscr{G}$ on P by automorphisms induces an action on E:

$$\mathscr{G} \times E \to E, \quad (\phi, [p, A]) \mapsto [\phi(p), A)]$$

If $[q, B] = [p, A]$, then there exists $g \in G$ such that $q = pg$ and $B = \mathrm{Ad}_{g^{-1}}(A)$, so

$$[\phi(q), B] = [\phi(pg), B] = [\phi(p), \mathrm{Ad}_g(B)] = [\phi(p), A].$$

This shows that the action of $\mathscr{G}$ on E is well–defined. ϕ maps ϕ^*H to H and so it maps the curvature $\widetilde{\Omega}$ of ϕ^*H to the curvature Ω of H, i.e.

$$\phi(\widetilde{\Omega}(X, Y)) = \Omega(X, Y) \qquad \forall X, Y \in T_m M$$

Choose$p \in \pi^{-1}(m)$ and write $\widetilde{\Omega}(X, Y) = [p, A]$ for some $A \in \mathfrak{g}$. Then

$$\phi\widetilde{\Omega}(X, Y) = [\phi(p), A] = [pu(p), A] = [p, \mathrm{Ad}_{u(p)}(A)]$$

So if $A \in \mathfrak{g}$ represents the curvature of ϕ^*H, then $\mathrm{Ad}_u(A)$ represents the curvature of H itself. If $\phi(p) = pu(p)$, then the curvature $\widetilde{\Omega}$ of ϕ^*H is $\widetilde{\Omega} = \mathrm{Ad}_{u^{-1}} \Omega$ where Ω is the curvature of H. (The same formula holds for the curvature as a 2–form).

Corollary. *If $\phi \in \mathscr{G}$ and H is a connection on P, then ϕ^*H is flat if and only if H is flat.*

Definition. Two connections H_1, H_2 on P are called gauge equivalent if there is a $\phi \in \mathscr{G}$ with $\phi^*H_1 = H_2$.

The corollary says that if H_1, H_2 are gauge equivalent, then H_1 is flat if and only if H_2 is flat.

Let $\langle -, - \rangle$ be a Riemannian metric on M. If M is oriented, then $\langle -, - \rangle$ induces a volume form dvol on M characterized by

$$\mathrm{dvol}(e_1, \ldots, e_n) = 1$$

if $e_1, \ldots, e_n$ is a positively oriented orthonormal basis for $(T_m M, \langle -, - \rangle)$. Assume that $\mathfrak{g}$ is equipped with an Ad–invariant, positive definite scalar product. $\langle -, - \rangle$ together with the Ad–invariant scalar product in $\mathfrak{g}$ induces a smooth fiber–wise metric on $\Lambda^2 T^*M \otimes E$.

Definition. If H is a connection on $P \to M$ with curvature Ω, where M is an oriented compact manifold, we define the *Yang–Mills–functional*

$$\mathcal{YM}(H) := \int_M \|\Omega\|^2 \,\mathrm{dvol}$$

Lemma. *$\mathcal{YM}(\phi^*H) = \mathcal{YM}(H)$, i.e. $\mathcal{YM}$ is $\mathscr{G}$–invariant.*

Proof. Let $\widetilde{\Omega}$ be the curvature of ϕ^*H. Since the scalar product on the fiber of E is Ad–invariant, we have

$$\|\widetilde{\Omega}\|^2 = \|\Omega\|^2$$

□

Remark. We have $\mathcal{YM}(H) \geq 0$ with equality if and only if H is flat.

Theorem. *There is a 1:1 correspondence between the flat connections on all possible principal G–bundles $P \to M$ up to gauge equivalence and the set* $\mathrm{Hom}(\pi_1(M), G)/G$ *where G acts on homomorphisms by conjugation.*

Lemma. *Let $P \xrightarrow{\pi} M$ be a principal G–bundle and H a connection on P. If H is flat, then P_c (with respect to H) depends only on the homotopy class of c.*

Proof. If H is flat, then by the Ambrose–Singer theorem, Hol_0 is trivial, so P_c is the identity if c is a closed loop which is null–homotopic through closed loops.

Let c_1, c_2 be paths from m_1 to m_2. Then c_1 travelled backwards followed by c_2 is a closed loop based at m_2. Parallel transport along this loop maps $P_{c_1}(p)$ to $P_{c_2}(p)$ for all $p \in \pi^{-1}(m_2)$. If c_1 and c_2 are homotopic with fixed endpoints, then this loop at m_2 is null–homotopic as a loop. Since H is flat, this implies $P_{c_1}(p) = P_{c_2}(p)$. □

If H is a flat connection on $P \xrightarrow{\pi} M$ and $p_0 \in \pi^{-1}(m_0)$, define the *holonomy representation*

$$\mathrm{hol}\colon \pi_i(M, m_0) \to G, \quad [\gamma] \mapsto g(\gamma)^{-1}$$

where $P_\gamma(p_0) = p_0 g(\gamma)$. This is well–defined by the lemma and is a group homomorphism $g(\gamma)g(\gamma') = g(\gamma'\gamma)$ since

$$p_0 g(\gamma)^{-1} g(\gamma')^{-1} = P_{\gamma'}(P_\gamma(p_0)) = P_{\gamma\gamma'}(p_0) = p_0 g(\gamma\gamma')^{-1}.$$

Suppose we use $p_1 \in \pi^{-1}(m_0)$ instead of p_0 to define hol. Then we get g_1 defined by $P_\gamma(p_1) = p_1 g_1(\gamma)$. There exists a unique $h \in G$ such that $p_1 = p_0 h$, so

$$p_0 g(\gamma) h = P_\gamma(p_0) h = P_\gamma(p_0 h) = p_0 h g_1(\gamma).$$

This implies $g_1(\gamma)^{-1} = h^{-1} g(\gamma)^{-1} h$, so the conjugacy class of hol is independent of the choice of basepoint $p_0 \in \pi^{-1}(m_0)$.

Lemma. *If H_1, H_2 are gauge equivalent flat connections on $P \xrightarrow{\pi} M$, then their holonomy representations are conjugate.*

Proof. Let $\phi\colon P \to P$ be a gauge transformation with $D\phi(H_1) = H_2$. Pick a basepoint $p_1 \in \pi^{-1}(m_0)$ to define hol_1, the holonomy representation of H_1. To define hol_2, the holonomy representation of H_2, use the basepoint $p_2 = \phi(p_1)$. Then $\mathrm{hol}_1([\gamma]) = g_1(\gamma)^{-1}$ and $\mathrm{hol}_2([\gamma]) = g_2(\gamma)^{-1}$ where g_1 and g_2 are defined by $P^1_\gamma(p_1) = p_1 \gamma_1(\gamma)$ and $P^2_\gamma = p_2 \gamma_2(\gamma)$ where P^1 is the parallel transport with respect to H_1 and P^2 is the parallel transport with respect to H_2. We have

$$p_2 g_2(\gamma) = P^2_\gamma(p_2) = \phi(p_1 g_1(\gamma)) = \phi(p_1) g_1(\gamma) = p_2 g_1(\gamma)$$

so $g_1(\gamma) = g_2(\gamma)$, meaning that $\mathrm{hol}_1 = \mathrm{hol}_2$ if we use p_1 respectively $p_2 = \phi(p_1)$ as basepoints for the definition of hol. □

Let $\rho\colon \pi_1(M, m_0) \to G$ be a representation. Consider the universal covering $\widetilde{M} \to M$. Definie $P\colon \widetilde{M} \times_\rho G = \widetilde{M} \times G/\simeq$ where

$$(x, g) \simeq (\gamma x, \rho(\gamma)g) \qquad \forall \gamma \in \pi_1(M, m_0)$$

P is the quotient of $\widetilde{M} \times G$ by the action of $\pi_1(M, m_0)$ given by

$$\pi_1(M, m_0) \times \widetilde{M} \times G \to \widetilde{M} \times G \quad (\gamma, x, g) \mapsto (\gamma x, \rho(\gamma)g)$$

The equivalence class of (x, g) is denoted by $[x, g]$. Define $\pi\colon P \to M, [x, g] \mapsto [x]$ where $\widetilde{M} \to M, x \mapsto [x]$ comes from the covering. This is well–defined and smooth. G acts on P on the right as follows:

$$P \times G \to P, \quad ([x, h], g) \mapsto [x, hg] =: [x, h]g$$

For any $\gamma \in \pi_1(M, m_0)$, we have $[x, h] = [\gamma x, \rho(\gamma)h]$. Since also

$$[\gamma x, \rho(\gamma)h]g = [\gamma x, \rho(\gamma)hg] = [x, hg]$$

this right G–action on P is well–defined.

The P constructed in this way from ρ is a principal G–bundle over M. $\widetilde{M} \times G$ has a natural connection whose horizontal subspaces are tangent spaces to $\widetilde{M} \times \{g\}$. This distribution is tautologically integrable, so the connection is flat. $\pi_1(M, m_0)$ acts on $\widetilde{M} \times G$ preserving the flat product connection, which therefore descends to P as a flat connection.

Suppose $\overline{\rho}$ is defined by $\overline{\rho}(\gamma) = \alpha\rho(\gamma)\alpha^{-1}$ for all γ and some fixed $\alpha \in G$. Then $\overline{\rho}$ gives rise to a principal G–bundle $\overline{P}$ with a flat connection $\overline{H}$.

Lemma. *There is an isomorphism of principal bundles $\phi\colon \overline{P} \to P$ with $D\phi(\overline{H}) = H$.*

Proof. We have $\overline{P} = \widetilde{M} \times G/\simeq$ with

$$(x, g) \simeq (\gamma x, \overline{\rho}(\gamma)g) = (\gamma x, \alpha\rho(\gamma)\alpha^{-1}g) \qquad \forall \gamma \in \pi_1(M, m_0)$$

Now define

$$\phi\colon \overline{P} \to P \quad [x, g]_{\overline{\rho}} \mapsto [x, \alpha^{-1}g]_\rho$$

This ϕ is well–defined since $[x, g]_{\overline{\rho}} = [\gamma x, \alpha\rho(\gamma)\alpha^{-1}g]_{\overline{\rho}}$ is mapped to $[x, \alpha^{-1}g]_\rho = [\gamma x, \rho(\gamma)\alpha^{-1}g]_\rho$. ϕ is also smooth. We have $\pi_P \circ \phi = \pi_{\overline{P}}$ and

$$\phi([x, g]_{\overline{\rho}}h) = \phi([x, gh]_{\overline{\rho}}) = [x, \alpha^{-1}gh]_\rho = [x, \alpha^{-1}g]_\rho h = \phi([x, g]_{\overline{\rho}})h.$$

This shows that ϕ is an automorphism of principal G–bundles. ϕ preserves the local product structures in which H, $\overline{H}$ are given by the tangent spaces to the first factor in $U \times G$, $U \subset M$ open. So $D\phi(\overline{H}) = H$. □

For the theorem, it remains to prove that the composition of these two maps. in both directions is the identity. Start with a representation $\rho\colon \pi_1(M, m_0) \to G$ and consider the corresponding principal G–bundle P and the flat connection H on P. Choose a basepoint $x_0 \in \overline{M}$ with $[x_0] = m_0$. Let γ be a loop in M based at m_0 Then γ has a unique lift $\widetilde{\gamma}$ to $\overline{M}$ with starting point x_0 and endpoint $[\gamma]x_0$.

Define $\overline{\gamma}(t) \coloneqq [\widetilde{\gamma}(t), e]$, where $e \in G$ is the neutral element. If $\pi\colon P \to M$ is the projection, then

$$\pi(\overline{\gamma}) = \pi([\widetilde{\gamma}(t), e]) = [\widetilde{\gamma}(t)] = \gamma(t)$$

so $\overline{\gamma}$ is a lift of γ from M to P. The starting point of $\overline{\gamma}$ is $[x_0, e]$. This lift is horizontal for the flat connection H because the curve $(\widetilde{\gamma}(t), e)$ in $\widetilde{M} \times G$ has tangent vector tangent to the first factors.

We use $p_0 = [x_0, e]$ as a basepoint in P to define hol.

$$\mathrm{hol}\colon \pi_1(M, m_0) \to G, \quad [\gamma] \mapsto g(\gamma)^{-1}$$

where $P_\gamma(p_0) = p_0 g(\gamma)$. Since $\overline{\gamma}$ is the unique horizontal lift of γ,

$$p_0 g(\gamma) = P_\gamma(p_0) = \overline{\gamma}(1) = [\widetilde{\gamma}(1), e] = [[\gamma]x_0, e] = [x_0, \rho([\gamma])^{-1}] = [x_0, e]\rho([\gamma])^{-1} = p_0 g(\gamma)^{-1}$$

so $\mathrm{hol} = \rho$.

Finally, start with a flat connection H on some principal G–bundle $P \xrightarrow{\pi} M$. Fix $p_0 \in \pi^{-1}(m_0)$ and define $\mathrm{hol}\colon \pi_1(M, m_0) \to G$ using the basepoint p_0. Define $\overline{P}$ by $\widetilde{M} \times G/ \simeq$ where $(x, g) \simeq (\gamma x, \mathrm{hol}(\gamma)g)$ for all $\gamma \in \pi_1(M, m_0)$. $\overline{P}$ has an obvious flat connection $\overline{H}$. We need to find an isomorphism $\phi\colon P \to \overline{P}$ with $D\phi(H) = \overline{H}$.

Let $H(p)$ be the holonomy bundle of $p \in P$. So $H(p) = \widetilde{M}/\Gamma$ where $\Gamma \subset \pi_1(M, m_0)$ is a subgroup. In fact, $\Gamma = \ker(\mathrm{hol})$. Define

$$\phi\colon H(p) \to \overline{P}, \quad [x] \mapsto [x, e]$$

If $\gamma \in \Gamma$, then $[x] = [\gamma x]$. Since

$$[\gamma x, e] = [x, \mathrm{hol}(\gamma)^{-1}] = [x, e],$$

ϕ is well–defined and smooth. If $q \in P \setminus H(p)$, there exists g such that $qg \in H(p)$. Define $\phi(q) \coloneqq \phi(qg)g^{-1}$. We will leave out the check that this is well–defined. This ϕ defined on all of P is then an isomorphism $P \to \overline{P}$ mapping H to $\overline{H}$.

Proposition. *Let φ be a $\mathfrak{g}$–valued 1–form on P satisfying $r_g^*\varphi = \mathrm{Ad}_{g^{-1}}\varphi$ and $\varphi(X) = 0$ if X is vertical. Then*

$$D\varphi(X, Y) = \mathrm{d}\varphi(X, Y) + [\omega(X), \varphi(Y)] + [\varphi(X), \omega(Y)] \qquad \forall X, Y \in TP$$

Proof. Both sides of the equation are bilinear and skew–symmetric. It suffices to check the three cases that X, Y are both horizontal, both vertical or one horizontal and the other vertical.

If both X and Y are horizontal, $\omega(X) = 0 = \omega(Y)$ and $D\varphi(X, Y) = \mathrm{d}\varphi(X, Y)$ by the definition of the covariant deriviative.

If X and Y are vertical, $\varphi(X) = 0 = \varphi(Y)$ and $D\varphi(X,Y) = 0$ since $\mathscr{H}X = 0 = \mathscr{H}Y$. Extend X, Y to fundamental vector fields $A^*_p = X$, $B^*_p = Y$, then

$$\mathrm{d}\varphi(A^*, B^*) = L_{A^*}(\varphi(B^*)) - L_{B^*}(\varphi(A^*)) - \varphi([A^*, B^*]) = 0$$

since $[A^*, B^*]$ is vertical and φ vanishes on fundamental vector fields, so $\mathrm{d}\varphi(X,Y) = (\mathrm{d}\varphi(A^*, B^*))(p) = 0$.

Given $X \in V_p$ and $Y \in H_p$ we choose extensions to vector fields on P as follows: X is extended by A^* with $A \in \mathfrak{g}$, such that $A^*_p = X$. Y is extended to a G–invariant horizontal vector field $\widetilde{Y}$ on P. This is possible since: $D_p\pi \colon H_p \to T_{\pi(p)}M$ is an isomorphism. We extend $D_p\pi(Y)$ to a vector field on M with support in a neighbourhood of $\pi(p)$ over which P is trivial. Choosing a section $s \colon U \to P$, U containing the support of the vector field in M, we can an isomorphism $D\pi \colon H_{s(U)} \to TU$. We lift the vector field on M under this isomorphism and use the G–action to extend it to a G–invariant horizontal vector field on P extending the original Y.

Now X is vertical, so $\varphi(X) = 0$ and $D\varphi(X,Y) = 0$ since $\mathscr{H}X = 0$. To check the claim in this case, we have to prove $\mathrm{d}\varphi(X,Y) = -[\omega(X), \varphi(Y)]$. ☐

Let ω_0 be a connection 1–form on P and ω is a 1–form on P with values in $\mathfrak{g}$ satisfying $r_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$ and $\omega|_V = 0$. Let $\omega_t = \omega_0 + t\omega$ with $t \in \mathbb{R}$. This is a smoothly varying familiy of connection 1–forms defining $H_t = \ker \omega_t$. Let Ω_t be the curvature of H_t. Then

$$\begin{aligned}\Omega &= \mathrm{d}\omega_t + [\omega_t, \omega_t] = \mathrm{d}\omega_0 + t\mathrm{d}\omega + [\omega_0 + t\omega, \omega_0 + t\omega] = \\ &= \mathrm{d}\omega_0 + [\omega_0, \omega_0] + t(\mathrm{d}\omega + [\omega_0, \omega] + [\omega, \omega_0]) + t^2[\omega, \omega] = \\ &= \Omega_0 + tD_0\,\omega + t^2[\omega, \omega]\end{aligned}$$

where D_0 is the derivative with respect to ω_0 or H_0.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_t\Big|_{t=0} = D_0\omega$$

With the Yang–Mills–functional

$$\mathcal{YM} \colon \mathrm{Conn}(P) \to R, \quad H \mapsto \int_M \|\Omega\|^2 \,\mathrm{dvol}$$

where we think of Ω as a section of $\Lambda^2 T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})$ and choose a Riemannian metric on M and an Ad–invariant scalar product on $\mathfrak{g}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(H_t)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \langle \Omega_t, \Omega_t\rangle \,\mathrm{dvol} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M \langle \Omega_0 + tD_0\omega + t^2[\omega,\omega], \Omega_0 + tD_0\omega + t^2[\omega,\omega]\rangle \,\mathrm{dvol} \\ &= 2\int_M \langle \Omega_0, D_0\omega\rangle \,\mathrm{dvol} = 2\int_M \langle D_0^*\Omega_0, \omega\rangle \,\mathrm{dvol}\end{aligned}$$

Proposition. $H_0 = \ker \omega_0$ *is a critical point of* $\mathcal{YM}$ *if and only if* $D_0^*\Omega_0 = 0$.

Remark. We always have $D_0\Omega_0 = 0$ by the Bianchi identity.

2.6 Principal S^1–bundles

We consider principal S^1–bundles over M. $\mathfrak{g}$–valued forms on P or M are ordinary forms. Because G is Abelian, $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ sends G to $\mathrm{id}_{\mathfrak{g}}$. A connection 1–form ω on a principal S^1–bundle $P \xrightarrow{\pi} M$ is an ordinary S^1–invariant 1–form on P, since

$$r_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega = \omega.$$

The curvature Ω is an S^1–invariant ordinary 2–form on P which vanishes on vertical vectors. We can think of Ω as an ordinary 2–form on M. By the structure equation

$$D\omega = \Omega = \mathrm{d}\omega + [\omega, \omega] = \mathrm{d}\omega$$

on P since G is Abelian. On M, Ω is closed but not neccessarily exact, because ω is not defined on M, only on P. In this case $D = \mathrm{d}$. The Yang–Mills–equation $D_0^*\Omega_0 = 0$ becomes $\mathrm{d}^*\Omega_0 = 0$. Since Ω_0 is closed, H_0 is a Yang–Mills–connection if and only if Ω_0 is a harmonic 2–form. Let ω_0, ω_1 be 2 different connection 1–forms on a principal S^1–bundle $P \to M$ and $\omega := \omega_1 - \omega_0$. Then

$$\Omega_1 = \mathrm{d}\omega_1 = \mathrm{d}(\omega_0 + \omega) = \mathrm{d}\omega_0 + \mathrm{d}\omega = \Omega_1 + \mathrm{d}\omega$$

with ω defined on M. So $[\Omega] \in H^2_{\mathrm{dR}}(M)$ is independent of the connection whose curvature we take.

Definition. $[\Omega] \in H^2_{\mathrm{dR}}(M)$ is the *Euler class* of $P \to M$ (or *first Chern class* if $G = U(1)$).

Given a principal S^1–bundle $P \xrightarrow{\pi} M$, let $C(P) \in H^2_{\mathrm{dR}}(M)$ be its Euler class and $\mathscr{C}_P$ the space of closed 2–forms on M whose cohomology class is $C(P)$.

Lemma. *Every $\alpha \in \mathscr{C}_P$ is the curvature of some connection on P.*

Proof. Choose some connection ω_0 on P with curvature Ω_0. Then $\Omega_0 - \alpha$ is exact and we can write $\alpha = \Omega_0 + \mathrm{d}\omega$ for some 1–form ω on M. Then $\omega_0 + \pi^*\omega$ is a connection 1–form with curvature α. □

Let $\mathcal{A}$ be the affine space of connections on P and define the map

$$c\colon \mathcal{A} \to \mathscr{C}_P, \quad \ker\omega = H \mapsto \Omega.$$

Then by the above lemma, c is surjective.

Lemma. *For every $\Omega \in \mathscr{C}_P$, the preimage $c^{-1}(\Omega)$ can be identified with $\mathcal{C}^1$, the space of closed forms on M.*

Proof. Let $H_0 = \ker\omega_0 \in c^{-1}(\Omega)$. Every other connection 1–form ω_1 on P is defined by $\omega_1 = \omega_0 + \pi^*\omega$ for some 1–form ω on M. The curvature Ω_1 of $H_1 = \ker\omega_1$ is

$$\Omega_1 = \mathrm{d}\omega_1 = \mathrm{d}\omega_0 + \mathrm{d}\pi^*\omega = \Omega + \pi^*\mathrm{d}\omega$$

so $H_1 \in c^{-1}(\Omega)$ if and only if $\pi^*\mathrm{d}\omega = 0$, which is equivalent to $\mathrm{d}\omega = 0$, i.e. $\omega \in \mathcal{C}^1$. □

The gauge group $\mathscr{G}$ is, since S^1 is Abelian,

$$\begin{aligned}\mathscr{G} &= \mathrm{Aut}(P) = \{u\colon P \to S^1 \mid u(pg) = g^{-1}u(p)g\} = \{u\colon P \to S^1 \mid u(pg) = u(p)\} \\ &= \{\overline{u}\colon M \to S^1\}.\end{aligned}$$

$\mathscr{G}$ acts on A by

$$\phi^*\omega_0 = \mathrm{Ad}_{u^{-1}}\,\omega_0 + u^*\theta = \omega_0 + u^*\theta = \omega_0 + \pi^*(\overline{u}^*\theta).$$

The curvature of $\phi^*\omega_0$ is

$$\mathrm{d}\phi^*\omega_0 = \mathrm{d}\omega_0 + \mathrm{d}\pi^*(\overline{u}^*\theta) = \mathrm{d}\omega_0 + \pi^*\overline{u}^*\mathrm{d}\theta = \mathrm{d}\omega_0.$$

The map $c\colon \mathcal{A} \to \mathscr{C}_P$ descends to

$$\mathcal{A}/\mathscr{G} \xrightarrow{\ \overline{c}\ } \mathscr{C}_P$$

which is again surjective and $\mathcal{C}^1$ surjects onto $\overline{c}^{-1}(\Omega)$.

Let $\mathscr{G}H_0 = \mathscr{G}\ker\omega_0 \in \overline{c}^{-1}(\Omega)$. Then every other gauge equivalence class in $\overline{c}^{-1}(\Omega)$ is represented by $\omega_0 + \pi^*\omega$ for some closed ω. What is the condition on ω and ω' to ensure that $\omega_0 + \pi^*\omega$ and $\omega_0 + \pi^*\omega'$ are gauge equivalent?

$$\omega_0 + \pi^*\omega' = \phi^*(\omega_0 + \pi^*\omega) = \omega_0 + \pi^*\omega + \pi^*(\overline{u}^*\theta) \iff \omega' = \omega + \overline{u}^*\theta$$

The map $\exp\colon \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ is a universal cover of S^1. We lift $\overline{u}\colon M \to S^1$ to $\widetilde{u}\colon M \to \mathbb{R}$ such that

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\widetilde{u}}{\nearrow} & \big\downarrow e^{2\pi i t} \\ M & \xrightarrow{\ \overline{u}\ } & S^1 \end{array}$$

commutes. So $\overline{u}^*\theta = \widetilde{u}^*\exp^*\theta = \widetilde{u}^*(\mathrm{d}t) = \mathrm{d}\widetilde{u}$, i.e. if $\overline{u} = \exp\circ\widetilde{u}$, then $\overline{u}^*\theta = \mathrm{d}\widetilde{u}$. Conversely, for every exact 1–form α on M, we can choose a $\widetilde{u} \in C^\infty(M)$ such that $\alpha = \mathrm{d}\widetilde{u}$ and consider $\exp\circ\widetilde{u} = \overline{u}$ as a gauge transformation of P. So $\mathcal{C}^1/\mathcal{E}^1 = H^1_{\mathrm{dR}}(M)$ surjects onto $\overline{c}^{-1}(\Omega)$.

$$[M, S^1] \to H^1(M, \mathbb{Z}), \quad [\overline{u}] \mapsto [\overline{u}^*\theta] = \overline{u}^*[\theta]$$

Lemma. *For any $\Omega \in \mathscr{C}_P$, the preimage $\overline{c}^{-1}(\Omega) \in \mathcal{A}/\mathscr{G}$ can be parametrized by the quotient*

$$H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}),$$

i.e. the sequence $\mathbb{C}^1 \to \mathcal{A} \to \mathscr{C}_P$ induces

$$H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) \longrightarrow \mathcal{A}/\mathscr{G} \longrightarrow \mathscr{C}_P.$$

A connection $H_0 = \ker \omega_0$ is a critical point of the Yang–Mills–functional if and only if $\mathrm{d}^*\Omega_0 = 0$. We know that $\mathrm{d}\Omega_0 = 0$. If M is compact without boundary, the pair of equations

$$\mathrm{d}\Omega_0 = 0 \qquad \mathrm{d}^*\Omega_0 = 0$$

Is equivalent to $\Delta\Omega_0 = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. By Hodge theory, there is a unique harmonic 2–form in $\mathscr{C}_P$. All Yang–Mills–connections on P map to the unique harmonic 2–form in $\mathscr{C}_P$.

The gauge equivalence classes of Yang–Mills–connections on P are parametrized by $H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$. If $H^1(M,\mathbb{R}) = 0$, then there is a unique gauge equivalence class of Yang–Mills–connections on every principal S^1–bundle $P \to M$.

2.7 The Yang–Mills–equations

Fix a Lie group G with the property that $\mathfrak{g}$ as a positive definite scalar product (e.g. G is compact). We fix once and for all such a $\langle -,- \rangle$. Let (M,g) be a compact Riemannian manifold, oriented without boundary.

If $P \xrightarrow{\pi} M$ is a principal G–bundle, then the space of connections $\mathcal{A}$ on P is an affince space for $\Omega^1(M,E) = \Gamma(T^*M \otimes E)$ where $E = P \times_{\mathrm{Ad}} \mathfrak{g}$. We have metrics on T^*M and E.

More generally, we can look at k–forms on M with values in E.

$$\Omega^k(M,E) = \Gamma(\Lambda^k T^*M \otimes E)$$

Example. The curvature form of a connection on P is an element of $\Omega^1(M,E)$.

Elements of $\Omega^k(M,E)$ correspond to k–forms α on P with the following properties:

(1) $\alpha(X_1,\dots,X_k) = 0$ if one of the $X_1,\dots,X_k$ is vertical.

(2) $r_g^*\alpha = \mathrm{Ad}_{g^{-1}}\,\alpha$ for all $g \in G$.

We define the following operations on $\Omega^k(M,E)$:

$$[-,-]\colon \Omega^k(M,E) \times \Omega^l(M,E) \to \Omega^{k+l}(M,E), \quad (\alpha \otimes v, \beta \otimes w) \mapsto (\alpha \wedge \beta) \otimes [v,w]_{\mathfrak{g}}$$

$$\wedge\colon \Omega^k(M,E) \times \Omega^l(M,E) \to \Omega^{k+l}(M), \quad (\alpha \otimes v, \beta \otimes w) \mapsto \langle v,w \rangle\, \alpha \wedge \beta$$

A connection $H = \ker \omega$ on P defines a covariant derivative D on $\mathfrak{g}$–valued k–forms on P by

$$D\alpha = \mathrm{d}\alpha \circ \mathscr{H}.$$

If α has values in $\mathfrak{g}$ and satisfies (1) and (2), then so does $D\alpha$. Therefore, D can be thought of as

$$D\colon \Omega^k(M,E) \to \Omega^{k+1}(M,E)$$

D is compatible with the metric and with $[-,-]$ and $\wedge$. That means: Let $V \to M$ be a vector boundle with scalar product $\langle -,- \rangle$. A covariant derivative $D\colon \Omega^k(M,V) \to \Omega^{k+1}(M,V)$ is compatible with $\langle -,- \rangle$ if

$$\mathrm{d}\langle s_1, s_2 \rangle = \langle Ds_1, s_2 \rangle + \langle s_1, Ds_2 \rangle.$$

For $\omega^k \in \Omega^k(M,E)$ and $\omega^l \in \Omega^l(M,E)$, we have

$$\mathrm{d}(\omega^k \wedge \omega^l) = D\omega^k \wedge \omega^l \pm \omega^k \wedge D\omega^l$$

Let V be a real vector space with an orientation, and $\langle -,-\rangle$ a positive definit scalar product on V. The volume form $\mathrm{dvol} \in \Lambda^n V^*$ where $n = \dim V$ is defined by $\mathrm{dvol}(e_1,\dots,e_n) = 1$ if $e_1,\dots,e_n$ is a positively oriented orthonormal basis of V. The scalar product induces a scalar product V^* by the requirement that the isomorphism $V \to V^*, v \mapsto \langle v,-\rangle$ should be isometric. This also gives rise to a scalar product $\langle -,-\rangle$ on $\Lambda^k V^*$ where vectors of the form $\lambda_{i_1} \wedge \dots \wedge \lambda_{i_k}$ with $\lambda_1,\dots,\lambda_n$ an orthonormal basis of V^* and $i_1 < \dots < i_k$ have length 1.

If $\langle -,-\rangle_0$ and $\langle -,-\rangle_1$ are two different scalar products on V, such that $\langle -,-\rangle_1 = \lambda^2 \langle -,-\rangle_0$ with $\lambda > 0$, then $\langle -,-\rangle_1 = \lambda^{-2k}\langle -,-\rangle_0$ on $\Lambda^k V^*$ and $\mathrm{dvol}_1 = \lambda^n \mathrm{dvol}_0$.

We define the Hodge operator

$$\star\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$$

by $\alpha \wedge \star\beta = \langle \alpha,\beta\rangle \mathrm{dvol}$ for all $\alpha,\beta \in \Lambda^k V^*$. One can check that $\star\star = (-1)^{k(n-k)}$.

If again $\langle -,-\rangle_1 = \lambda^2 \langle -,-\rangle_0$ on V, then

$$\alpha \wedge \star_1 \beta = \langle \alpha,\beta\rangle_1 \,\mathrm{dvol}_1 = \lambda^{-2k}\langle \alpha,\beta\rangle_0 \lambda^n \,\mathrm{dvol}_0 = \lambda^{n-2k}\alpha \wedge \star_0 \beta$$

for all $\alpha,\beta \in \Lambda^k V^*$, so $\star_1 = \lambda^{n-2k}\star_0$ on $\Lambda^k V^*$.

Let M be an oriented Riemannian manifold. Then the Hodge star

$$\star\colon \Omega^k(M) \to \Omega^{n-k}(M)$$

where $n = \dim M$ is defined fiberwise as above, i.e.

$$\alpha \wedge \star\beta = \langle \alpha,\beta\rangle \,\mathrm{dvol} \qquad \star\star = (-1)^{n(n-k)} \qquad \forall \alpha,\beta \in \Omega^k(M)$$

Let $P \to M$ be a principal G–bundle, fix an Ad–invariant positive definit scalar product on the Lie algebra $\mathfrak{g}$ and define $E := P \times_{\mathrm{Ad}} \mathfrak{g}$. We can extend $\star$ to a map $\Omega^k(M,E) \to \Omega^{n-k}(M,E)$ using the same formula with the above defined operation $\wedge\colon \Omega^k(M,E) \times \Omega^{n-k}(M,E) \to \Omega^n(M)$.

Lemma. *Let M be a compact oriented Riemannian manifold without boundary. On $\Omega^k(M,E)$, $k \geq 1$, we have $D^* = (-1)^{nk-n+1} \star D\star$, where D is the covariant derivative defined b a connection on P.*

Proof. Let $\alpha \in \Omega^{k-1}(M,E)$ and $\beta \in \Omega^k(M,E)$. Then by Stokes' theorem,

$$\begin{aligned} 0 &= \int_M \mathrm{d}(\alpha \wedge \star\beta) = \int_M D\alpha \wedge \star\beta + (-1)^{k-1}\int_M \alpha \wedge D\star\beta \\ &= \int_M \langle D\alpha,\beta\rangle \,\mathrm{dvol} + (-1)^{k-1}\int_M \langle \alpha, \star^{-1} D\star\beta\rangle \,\mathrm{dvol} \end{aligned}$$

So we have, for $\beta \in \Omega^k(M,E)$:

$$D^*\beta = -(-1)^{k-1}(-1)^{(k-1)(n-k+1)} \star D\star\beta = (-1)^{nk+n+1} \star D\star\beta \qquad \square$$

Now for $\omega_t = \omega_0 + t\omega$ with $t \in \mathbb{R}$ and $\omega \in \Omega^1(M, E)$, if ω_0 is a critical point of the Yang–Mills–functional, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{YM}(\omega_t)\Big|_{t=0} = 2\int_M \langle \Omega_0, D_0\omega\rangle \,\mathrm{dvol} = (-1)^{n+1} 2\int_M \langle \star D_0 \star \Omega_0, \omega\rangle \,\mathrm{dvol}.$$

So

$$D_0 \star \Omega_0 = 0$$

which is the *Yang–Mills–equation* for ω_0. The Yang–Mills–equation is a second order differential equation.

Assume $n = \dim M = 4$. Then $\star\Omega_0$ is a 2–form just like Ω_0 itself. By the Bianchi identity, $D_0\Omega_0 = 0$. We want to know when $\Omega_0 = \star\Omega_0$ holds. In the case $n = 4$, $k = 2$, $\star$ is an endomorphism of the 6–dimensional vector space $\Lambda^2 V$. It has eigenvalues ± 1 and this splits $\Lambda^2 V$:

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

where $\star$ acts as $\pm$ on $\Lambda^2_\pm V$. $\Lambda^2_+ V$ is called the *self–dual (SD)* and $\Lambda^2_- V$ the *anti–self–dual (ASD)* part of $\Lambda^2 V$.

If $\alpha_1, \ldots, \alpha_4$ is an orthonormal basis for V^*, then we have a basis for $\Lambda^2_\pm$ given by

$$\begin{aligned} &\alpha_1 \wedge \alpha_2 \pm \alpha_3 \wedge \alpha_4 \\ &\alpha_1 \wedge \alpha_3 \pm \alpha_4 \wedge \alpha_2 \\ &\alpha_1 \wedge \alpha_4 \pm \alpha_2 \wedge \alpha_3 \end{aligned}$$

If $\omega \in \Lambda^2_+ V$ and $\eta \in \Lambda^2_- V$, then

$$\omega \wedge \eta = \star\omega \wedge \eta = \eta \wedge \star\omega = \langle \eta, \omega\rangle \,\mathrm{dvol} = 0$$

$\Lambda^2_\pm V$ are orthogonal for $\langle -, -\rangle$ and for $\wedge$. If a connection ω_0 has self–dual or anti–self–dual curvature, then $\star\Omega_0 = \pm\Omega_0$ and $D_0 \star \Omega_0 = 0$ because of the Bianchi identity.

The equation $\star\Omega_0 = \pm\Omega_0$ is the *(anti–)self–duality equation* for ω_0. It implies the Yang–Mills equation.

Lemma. *On a 4–manifold M, the Yang–Mills–equation and the (anti–)self–duality equation are conformally invariant.*

Proof. Two metrics $\langle -, -\rangle_1$ and $\langle -, -\rangle_0$ on M are conformally invariant if $\langle -, -\rangle_1 = \lambda^2 \langle -, -\rangle_0$ for some $\lambda \neq 0$, $\lambda \in C^\infty(M)$.

We calculated that on k–forms, $\star_1 = \lambda^{n-2k}\star_0$, so in our case $\star_1 = \star_0$ for 2–forms and so the (A)SD equations for 2–forms with respect to the two metrics agree. The Yang–Mills equation is $D_0 \star \Omega_0 = 0$. Since the metric enters only in $\star$, this is the same for both metrics. □